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Dr. J. L. E. DREYER, PRESIDENT, in the Chair.

A. J. R. Atkin, Department of Mines, P.O.B. 6231, Johannesburg, South Africa;
William Fraser, 212 Causewayside, Edinburgh;
Lieut. Joseph Alfred Heenan, R.N.R., Dixie Lodge, Sandown Park, Wavertree, Liverpool;
Edwin Henry Frank Lawrence, 2nd Officer, Mercantile Marine, Hollydene, 18 Waterloo Road, Southampton, Hants; and
Rev. Daniel J. McHugh, C.M., Professor of Astronomy, De Paul University, 1010 Webster Avenue, Chicago, Illinois, U.S.A.,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Rev. Herbert Douglas Anthony, The College, Richmond, Surrey (proposed by H. Spencer Jones);
K. Chand, B.A., Sidney Sussex College, Cambridge (proposed by A. S. Eddington);
E. C. Isaac, 47 Apuka Road, Brooklyn, Wellington, New Zealand (proposed by C. E. Adams);
Rev. William Lipscombe Orpwood, St. Matthew's Vicarage, Hurlingham, London, S.W. (proposed by M. V. de Rivas);
Rev. Arthur Maitland Peckham, F.Z.S., 28 Hornsey Rise, London, N. 19 (proposed by A. R. Johnson);
Harold Woodgate Raisin, B.Sc., 24 Arnewood Road, Bournemouth (proposed by P. E. Vizard); and
Professor Issei Yamamoto, Kyoto University Observatory, Japan (proposed by J. A. Parkhurst).

Sixty-eight presents were announced as having been received since the last Meeting, including, amongst others :—

Ephemerides, 1557 (presented by Mr. A. Berry);
Schroeter (J. Fr.), Sonnenfinsternisse von 600 bis 1800 N. Chr.
Lowell Observatory, Five transparencies of Jupiter and Saturn, photographed with the 24-inch Refractor (presented by Professor E. C. Slipher);
J. H. Worthington, Four transparencies and prints of the total solar eclipse of 1911 April 28.

The President delivered a short Address, after which the Jackson-Gwilt Medal and Gift were handed to Mr. W. S. Franks, to whom the Medal had been awarded for his work on star colours and observations of double stars, and to the Rev. T. E. R. Phillips for transmission to Mr. A. Stanley Williams, to whom the Medal had been awarded for his work on planets and variable stars.

Addresses delivered by the President on presenting the Jackson-Gwilt Medal and Gift to Mr. A. Stanley Williams for his work on Planets and Variable Stars, and to Mr. W. S. Franks for his work on Star Colours and Observations of Double Stars.

Mr. Williams' observations of Jupiter began in 1879, the year in which the great red spot first attracted attention. Using at first a small refractor he became in the following year the possessor of a $5\frac{1}{4}$-inch reflector, with which he commenced regular work. Soon after that a $6\frac{1}{2}$-inch reflector was obtained, with which Mr. Williams has steadily and without interruption followed Jupiter year after year. His observations are undoubtedly made with extreme care, and though a micrometer has not been employed, the method of estimating the time when a marking is in mid-transit across the disc is certainly capable of producing valuable results, when used by a skilful observer throughout a long period of time. Mr. Williams has published two volumes of *Zenographical Fragments*, recording his observations of the oppositions of 1885–86 and 1888, but otherwise nearly all his results have been published in a long series of papers in the *Monthly Notices* from vol. lv. and down to the present time. Of all the complicated appearances of the surface of Jupiter seen during the last forty-three years there can hardly have been one which has not been regularly observed and discussed by Mr. Williams. In vol. lvi. is perhaps his most important paper, "On the Drift of the Surface Material of Jupiter in Different Latitudes," in which he made use of observations by various observers from Schröter down to his own time. The great equatorial current and the periodical

variation in colour of the equatorial regions are among the more important phenomena studied by Mr. Williams, and the value of this long continued series of observations must be very great. Mr. Williams has been blessed with exceptionally keen eyesight for planetary features, and in this kind of work much more depends on the man than on the telescope. It is really quite remarkable what he has been able to do with so modest an aperture as 6½ inches. Even many delicate linear markings on Mars were seen by him, and so long ago as 1890 he was able to confirm many of Schiaparelli's observations.

Mr. Williams has also paid some attention to Saturn and determined the rotation period in 1891–93. But the matter forming the surface of this planet is in a far more mobile state than that of Jupiter, and the determination of the period of rotation in different latitudes is therefore much more uncertain, while the opportunities of studying the surface markings are comparatively rare.

Of late years Mr. Williams has divided his attention between Jupiter and variable stars. He had already, in 1885, made a study of stellar magnitudes in the southern hemisphere, taking advantage of a voyage to Australia and a sojourn of two months in that continent. This resulted in *A Catalogue of the Magnitudes of* 1081 *Stars lying between* −30° *Declination and the South Pole* (London 1898). Since about the year 1900 he has observed and determined the elements of a small number of short-period variables and some Algol stars. He has also with a 4·4-inch portrait lens taken photographs of various parts of the heavens and in this way he has discovered several new variables, including some interesting objects of the Cepheid type, which he has then observed visually, thereby materially helping to increase our knowledge of this class of objects.

I regret that Mr. Williams has not been able to be present here to-day, but will ask Mr. Phillips (who is himself a successful worker in the same fields) to forward this medal and gift, presented by the Council to Mr. Williams in recognition of his long-continued valuable observations.

Mr. Franks was induced to take up the study of star colours by the late Mr. T. W. Webb, author of *Celestial Objects*, in 1877, and in 1878 June he presented to the Society a catalogue of the colours and magnitudes of 3890 naked-eye stars between the North Pole and −25° declination, from observations made with a 5-inch refractor. Most of this was published in abbreviated form in the *Harvard Photometry* (*Annals*, vol. xiv.). During the next ten years, while living at Leicester, Mr. Franks continued to make colour observations, and in 1886 March he completed a second catalogue of 1730 stars down to −20° Declination, brighter than the sixth magnitude, from observations made in the course of a year and a half with two Newtonian reflectors of 11¼ and 5¼ inches aperture. The introduction was printed in the *Monthly Notices* for 1886 April, and both these catalogues were deposited in our library. By this time Mr. Franks felt that the vague and indefinite terms hitherto used in naming the colours of stars, comparing them to the tints of flowers, fruit, or precious stones, were most unsatisfactory.

He therefore devised the system of notation described in the *M.N.* for 1887 March, which is suitable for use in discussing a number of observations. It was at once brought into use in a fresh survey of 758 of the brighter stars, employing the two reflectors already used, as well as a third, of 8½ inches aperture. Each star was observed on three different occasions, and the "mean colour" deduced from the observations.

For some years Mr. Franks worked at Mr. Isaac Roberts' observatory and had to devote himself to work of a different kind. But after the death of Mr. Roberts he resumed work on star colours, and in 1907 June he published the results of an investigation of the relation between the colours and spectra of 1360 bright stars in both hemispheres, the colour depending on observations made by members of the B.A.A. in England and Australia. In the following year he carried this inquiry a stage further, comparing his own MS. synoptic catalogue of 4175 stars (including all his own previous work) with the *Revised Harvard Photometry*. On the whole there is a striking agreement between the colours and spectra of stars, though there are some curious exceptions.

Ten years ago Mr. Franks became observer at Mr. F. J. Hanbury's Observatory, Brockhurst, East Grinstead, Sussex. He has been chiefly occupied in measuring very wide double stars, the distances ranging from 2′ down to 5″ or 6″, and on a few occasions even less. The instrument used is a 6-inch refractor by Cooke, and the observations were made in order to supply material for the determination of proper motions. The results have been published in eight annual catalogues in the *Monthly Notices*. But the star colours were not forgotten. The *Index Catalogue*, compiled by the late F. Krüger and published by Father Hagen in 1917, gave in a separate column the mean results of Mr. Franks' numerous observations. Finally he was asked by Father Hagen to undertake a revision of his observations in the *Index Catalogue* (column Fr.) and to fill in the gaps relating to stars of smaller magnitude, which were not included in his previous lists. This revision has been carried out with the 6-inch refractor, and the final results of Mr. Franks' work of many years, rounded off and completed, have quite recently been published as No. 15 of the Vatican Observatory Publications, containing 6611 stars.*

Mr. Franks, I have much pleasure in handing you this medal and gift as a token of our appreciation of your work.

* "The Letter- and Number-Scales of Visual Star Colours," by W. S. Franks and J. G. Hagen, S.J. (Rome, 1923).

The Problem of Electron-capture in the Stars. By A. S. Eddington, F.R.S., Plumian Professor.

In former papers * I have shown that the calculation of the absorption of radiation in a star raises the problem of capture of free electrons by atoms. The hypothesis that capture occurs by collision with the nucleus was considered, and it was found that this gives results in accordance with astronomical data. The chief object of the present paper is to give a simpler and more general derivation of the condition that a free electron shall collide with the nucleus; a general discussion of the problem of electron-capture is added, with some reference to various criticisms brought against the theory I have suggested.

1. Consider an electron with initial velocity V which would, if its path were undisturbed, pass the nucleus at minimum distance σ. Owing to the attraction of the nucleus the electron is drawn in and actually passes at a smaller distance d. The result is that the electrons approaching within distance d are those aimed at an apparent target of area $\pi\sigma^2$ and their number is proportional to this area. We have to find the formula connecting σ^2 and d. In my former paper I found it by tracing the whole orbit of the electron. I give here a simpler deduction, (*a*) without correction for change of mass with velocity, (*b*) including the correction.

(*a*) At nearest approach d the velocity V′ is at right angles to the radius vector, so that conservation of angular momentum gives

$$V'd = V\sigma \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Conservation of energy gives

$$\tfrac{1}{2}mV'^2 = \tfrac{1}{2}mV^2 + \epsilon\phi_d$$

where ϕ_d is the potential at distance d, and ϵ is the charge of the electron. In astronomical applications $\tfrac{1}{2}mV^2$ is of order 700 volts, and is small compared with $\epsilon\phi_d$ at the boundary of the nucleus which is of order 5,000,000 volts. Hence very approximately

$$\tfrac{1}{2}mV'^2 = \epsilon\phi_d \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2)

$$\sigma^2 = d^2\phi_d\frac{2\epsilon}{mV^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the inverse-square law $\phi_d = N\epsilon/d$ where N is the atomic number of the nucleus. Writing $b = \epsilon^2/mc^2 = 2{\cdot}81 \,.\, 10^{-13}$ cm. (3) becomes

$$\sigma^2 = 2Nbdc^2/V^2 \,. \quad . \quad . \quad . \quad . \quad (4)$$

which is the required formula.

(*b*) The mass of the electron at distance d is βm, and its momentum is $\beta mV'$ where $\beta = (1 - V'^2/c^2)^{-\frac{1}{2}}$. The gain of mass is thus $(\beta - 1)m$. The energy is c^2 times the mass, so that the gain of energy is $(\beta - 1)mc^2$

* *Monthly Notices*, 83, 32, 98.

which must be equated to $\epsilon\phi_d$. (We neglect as before the small initial kinetic energy of the electron.) Hence in place of (1) and (2) we have

$$\beta V' d = V\sigma \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$(\beta - 1)mc^2 = \epsilon\phi_d \quad . \quad . \quad . \quad . \quad (2a)$$

By (2a)

$$\beta = 1 + \epsilon\phi_d/mc^2.$$

Also

$$V'^2/c^2 = 1 - 1/\beta^2,$$

so that

$$\frac{\beta^2 V'^2}{c^2} = \beta^2 - 1 = \frac{2\epsilon\phi_d}{mc^2}\left(1 + \tfrac{1}{2}\frac{\epsilon\phi_d}{mc^2}\right).$$

Hence by (1a)

$$\sigma^2 = d^2\phi_d \frac{2\epsilon}{mV^2}\left(1 + \tfrac{1}{2}\frac{\epsilon\phi_d}{mc^2}\right) \quad . \quad . \quad . \quad (3a)$$

With the inverse-square law the extra factor is equal to $(1 + \frac{1}{2}Nb/d)$, so that (4) is replaced by

$$\sigma^2 = 2Nb(d + \tfrac{1}{2}Nb)c^2/V^2 \quad . \quad . \quad . \quad . \quad (4a)$$

It may be noted that the neglect of $\frac{1}{2}mV^2$ in comparison with $\epsilon\phi_d$ is equivalent to the approximation $e = 1$ in my former paper (p. 36).

2. The present proof calls attention to the fact that the behaviour of the electron in the neighbourhood of the nucleus is conditioned solely by the initial angular momentum σV, and does not depend on the initial energy $\frac{1}{2}mV^2$ which is unimportant. Thus the effect of varying V is equivalent to varying the limits of $\pi\sigma^2$ in the ratio V^{-2}, so that the number of electrons accomplishing any particular task (*e.g.* approaching within given distance, or hitting the nucleus at given angle) is proportional to V^{-2}. If, then, capture occurs in the neighbourhood of the nucleus we must expect the probability to be proportional to V^{-2}, or inversely proportional to the temperature. I have shown (*loc cit.*, p. 99) that this leads immediately to the law of absorption $mkT^{\frac{7}{2}}/\rho = \text{const.}$

This suggests that the stellar absorption law ought to provide a crucial test as to whether capture occurs near the nucleus where $\frac{1}{2}mV^2$ is negligible, or in the outer regions of the atom where it is important. This conclusion, however, is scarcely justified. The electron must lose energy greater than $\frac{1}{2}mV^2$ in order to be captured, so that although the initial energy has no share in determining what takes place near the nucleus, it may determine whether what takes place is sufficient to result in capture.

The formulæ (3) and (3a) do not assume the inverse-square law of force, so that the systematic shielding of the nucleus by the negative charge of the satellite electrons may be taken into account. The effect is to reduce ϕ_d very slightly, but it is evident that the alteration is insignificant. Accidental shielding due to the discontinuous distribution of this negative charge cannot be included because it disturbs the angular momentum, but it does not seem possible that this can have any effect on the average at the velocities here considered.

3. It was explained in my first paper that theoretical considerations do not decide whether formula (4) or (4*a*) is likely to be nearer the truth. I learn, however, that comparison of theory and measurement for the wave-lengths of the hydrogen spectrum establishes to a high degree of approximation the hypothesis underlying the latter formula, viz. that the mass of an electron moving in the field of the nucleus increases with velocity in the ratio β. This evidence seems to be decisive, and I accordingly adopt (4*a*) as correct. It gives a somewhat higher value of the absorption-coefficient, and this is confirmed by calculation for Capella—the only star for which accurate data are available (*loc. cit.*, p. 108).

For an element of average weight, $\frac{1}{2}Nb$ is considerably larger than the estimated value of d, so that as a rather rough approximation (4*a*) becomes

$$\sigma = N^2 b^2 (c/V)^2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The condition of capture by the nucleus does not involve to any considerable extent the actual size of the nucleus. The laws of absorption will thus be brought into line with laws of emission in which the nucleus is represented only by its mass and atomic number without further characterisation. On the other hand, the picturesque possibility of determining the size of the atomic nucleus from observations of the stars is abandoned.

To those who object to the hypothesis of nuclear capture it may be conceded that there is a greater chance of obtaining nearly equivalent formulæ by other mechanisms of capture now that d has practically disappeared from the results.

In my first paper I considered an alternative mode of capture, viz. that the electron loses energy by continuous radiation arising from its acceleration. The result was (p. 43, equation (20))

$$\sigma^2 = (4\pi)^{\frac{2}{5}} N^{\frac{2}{5}} b^2 (c/V)^{\frac{14}{5}} \quad . \quad . \quad . \quad . \quad (6)$$

Thus the ratio of the number of captures on the two hypotheses is

$$\left(\frac{4\pi}{N}\right)^{\frac{2}{5}} \left(\frac{c}{V}\right)^{\frac{4}{5}}.$$

The first factor can make no great difference; in stellar problems c/V is about 20, so that the opacity on the second hypothesis is considerably higher, and perhaps outside the limits which can be reconciled with astronomical data. But the two formulæ would not be easily distinguished if applied to the problem of capture of (or production of spectrum by) β rays or high-speed cathode rays. *E.g.* for cathode rays at 40,000 volts absorbed in platinum the two formulæ become equivalent.

Probably the formula (6) would only be accepted in modern atomic physics if modified so as to replace the continuous radiation by quantum emission, the continuous formula being in some sense the limit of the true formula. Such a theory may perhaps be encouraged by the result that the limiting expression (6) is not far from the order of magnitude required.

4. I find it difficult to understand the ground of the criticism made by Professor Lindemann and others that my view of the progress of the electron through the atom is at variance with the modern physical ideas. It is well known that atomic problems are treated nowadays by an incongruous mixture of classical and quantum dynamics; and every theory is liable to the criticism that it leans too much to one side or the other of the balance. But the consideration of the passage of a free electron through an atom is now an everyday problem of physics; and I have been guided by the leading investigations of this problem by Bohr, Rutherford, Jeans, Richardson, and others. They all trace the path of the particle by classical dynamics, with results often in reasonable accordance with observation. The luck may break down some day, but we may trust it as long as it lasts. Lindemann anticipates some kind of upheaval of classical dynamics as the electron passes the K and L levels,* but the authors quoted above do not entertain the idea.

Consider a small pencil of electrons falling centrally on an atom. The outer members are supposed by physicists to be deflected by classical laws, and scattering experiments are believed to confirm the predicted deflections. We can trace paths nearer to the axis until we reach paths which actually intersect the nucleus.† My idea is that these are real paths of the corresponding electrons. It is absurd to suppose that an electron entering one of these paths has a premonition of its fate, and jumps aside earlier in the path to avoid it. It is not until the electron is almost on the nucleus that its circumstances differ in any way from those of the scattered electrons. If physicists are justified in applying the classical laws to the latter, the same laws should be applicable to the former.

It is difficult to see how an electron approaching close to the complicated system of charges constituting the nucleus can avoid imparting some of its energy to it. The rigidity of the nucleus is very great, as is evidenced by the frequency of the oscillations shown by the γ rays; but the electron is approaching with nearly the speed of light and its field is concentrated into a thin pulse of width much less than the wave-length of γ rays. We cannot feel certain what will be the laws of interaction between the electron and the system of the nucleus, but there seems to be at any rate an opportunity for a transfer of energy. Admitting that there is some loss of energy at collision, this loss is scarcely likely to be less than the 1/10,000th part required for capture in astronomical problems. The high-speed electrons of laboratory experiments may perhaps be less easily captured.

Accordingly the electrons indicated by our formula (4*a*) run at any rate a grave risk of capture; and according to astronomical data this is approximately the number captured in stellar conditions. On present evidence it seems gratuitous to devise a mechanism for saving these

* *Monthly Notices*, 83, 333.

† The idea sometimes held that there is a region of static repulsive force protecting *but not forming part of* the nucleus is self-contradictory. Any region where the electrostatic potential does not conform to $\nabla^2\phi=0$ is *ipso facto* occupied by electric charge.

particular electrons and substituting an equal number of victims from other parts of the atom.

5. Professor Lindemann seems to consider that on my theory the electron must at the nuclear encounter not merely lose more than the 700 volts energy (required for capture), but must lose the exact amount which will bring it to a particular quantum orbit. This cannot be essential. In the quantisation of the atom the energy forms a common stock not apportioned to particular electrons; or, if we insist on apportioning it, it is necessary to concede that the atom has the power of distributing to each electron the energy needed for its assigned orbit. For example, when an electron in the L ring falls into the K ring to fill a vacant place, all the electrons left in the L ring must be reorganised in new orbits with changed energies; the energy radiated is not the energy given up by the particular electron which fell, but is the surplus energy of the system as a whole. Similarly in nuclear capture the energy radiated is not the energy lost by the electron in colliding with the nucleus, but is the surplus left in the whole atom when the final steady state is established. So far as my theory is concerned the problem ends when the electron has been shown to be captured; the subsequent readjustment of the system (now containing an additional member) to a possible steady state may be expected to proceed in the same manner as in other cases of disturbance of the atomic system.

Lindemann also takes the view that the target for capture should be the area of the ionised atom—about 10,000 times as large as the target given by my formula (*Observatory*, **45**, 384). If the free electrons were absorbed so rapidly I do not see how the electron-gas could ever reach temperature equilibrium with the atom-gas; it is well known from thermionic experiments that such equilibrium is actually attained.

To provide the increased number of free electrons to be captured by this larger target Lindemann supposes that the electrons are liberated mainly by collisions of the electron-gas with the atoms, only a very small proportion being liberated by absorption of radiation. I think that energy considerations will show that this is impossible. The average energy of a free electron is $\frac{3}{2}$RT, and we may agree with Lindemann that the atoms are ionised down to about the corresponding negative energy-level, $-\frac{3}{2}$RT.* Consequently the average emission of energy per capture is about 3RT; in no case is it less than $\frac{3}{2}$RT, the amount given up when an electron with no energy drops into a vacant place at the ionisation level. The emission of energy per second is equal to the absorption, hence the total possible number of captures is limited by the total amount of radiation absorbed. The average energy of a quantum is 2·7 RT; thus for each quantum absorbed there is only about enough energy for one capture, and certainly not enough energy for more than two captures. Lindemann's 10,000 captures per quantum are out of the question.

There are two ways by which the number of captures can be deduced from the amount of absorption, (1) by equating the number of electrons liberated by absorption to the number captured, (2) by equating the

* This is only a rough value, since the ionisation depends on the pressure as well as the temperature.

energy absorbed to the energy liberated on capture. The first method is followed in my paper, but it is noted (p. 39, *footnote*) that the second method checks the result approximately. I should be very glad of data which might enable an estimate to be made of the number of additional electrons set free by collision, so as to take account of the necessary correction if appreciable; but the foregoing considerations show that the correcting factor is limited to a value between 1 and 2.

The Otto Struve (OΣ) *Double Stars.* By J. Jackson, M.A., B.Sc.

(*Communicated by the Astronomer Royal.*)

Introduction.—The greater number of the OΣ stars were discovered by Otto Struve between 1841 August and 1842 December, with the 15-inch refractor of the Pulkowa Observatory. The principal instruments of the observatory which had recently been completed were a meridian circle and the refractor, which was by far the most powerful instrument of that time. In the formation of a working catalogue for the meridian instrument, about 17,000 stars in the northern hemisphere, down to the 7th or 8th magnitude were examined, and 514 double stars not included in the Dorpat Catalogue of W. Struve (Σ) were found. Of these 37 had previously been discovered in England, mostly by Sir John Herschel, but they were included in the new catalogue. Other double stars discovered at Pulkowa up to about 1870 were later added to the catalogue, and in all there are 547 stars which have been catalogued as OΣ stars.

Most of these stars were measured both by Otto Struve and Maedler (at Dorpat) before 1850, and by Dembowski between 1860 and 1870. They have been regularly measured by later double-star observers, so that we now have measures covering a period of about three-quarters of a century. It is therefore possible in most cases to form a definite estimate of the importance of the different systems. This is greatly facilitated by the work of Hussey who, in the closing years of the last century, reobserved all the OΣ stars with the 12-inch and 36-inch telescopes of the Lick Observatory.

It was originally intended that the limiting separation should be 32″, and that this should be reduced to 16″ when the companion was fainter than the 9th magnitude. Sixty-five stars included at first but afterwards found not to satisfy these conditions were rejected, leaving only 15 stars with separation 16″–32″. The great importance of very close pairs led Otto Struve to include a considerable number of stars which have since proved to be single. Twenty-two of these he himself rejected later, and Hussey rejected 14 others: 3 of the 22 rejected by OΣ are now known to be double, viz. OΣ 61, 192, and 271. Otto Struve rejected other 19 stars for various reasons, 10 of these were Σ stars. Other 5 are now recognised as Σ stars. OΣ 21 is here also rejected, but it has been observed recently as a very close pair (separation 0″·2) by Aitken. We

therefore reject in all 123 stars, leaving 424 to be considered, of which OΣ 425 is a triple star counted twice in the following discussion.

As a class the OΣ stars are considerably different from the Σ stars. The following figures show the greater proportion of close pairs.

	Number of Stars.		Percentage of Stars.	
	0″–1″.	1″–2″.	0″–1″.	1″–2″.
Σ Catalogue of 2640 stars	86	317	3·3	12·0
OΣ Catalogue of 547 stars	147	69	26·9	12·6

Another reason why many OΣ stars are difficult to observe is that there is often a great difference in the brightness of the components. It is sometimes thought that a double star composed of stars of very unequal brightness will generally prove to be merely an optical system, but although most optical systems are of this type the total number of close optical systems is not large. Several of the best-known binaries, *e.g.* Sirius and Procyon, are composed of stars of very different brightness, and there is no reason to suppose that such systems are rare. Consider, for example, the triple system Σ 1516 and OΣ 539. The magnitudes of the three stars are A = 7·0, B = 7·5, C = 12·0. A and B form Σ 1516; they were separated by about 10″ in 1831, but the separation decreased to about 2″·3 in 1855, and it is now about 30″. This is an optical system, and the change is due to the proper motion of A. The faint star C is moving with A and slowly revolving round it at a distance of about 8″. This system shows the danger of deciding whether a system is optical or binary simply from the brightness of the components.

Classification.—An examination of the OΣ stars has led to the following classification, similar to that given for the Σ stars (*M.N.*, 83, 4):—

A. Pairs showing relative motion:
- I. Binary systems, 135.
- II. Optical systems, 15.
- III. Nature uncertain, 5.

B. Pairs showing no certain relative motion:
- IV. Close pairs, 227.
- V. Intermediate pairs, 34.
- VI. Wide pairs, 9.

In Class IV. the separation is under 5″ for pairs in which the brighter component is of magnitude 9·0: for brighter stars an additional factor of $10^{0·2}$ or 1·58 is allowed in the separation, for each magnitude. In Class V. twice the separation of Class IV. is allowed, and Class VI. contains wider pairs. Class IV. includes a few fairly wide pairs in which the brighter star is of the 3rd or 4th magnitude.

Binaries.—Table I. contains the OΣ numbers of the binary systems in motion. Hypothetical parallaxes for 101 of these have already been given by Mr. Furner and myself in *Monthly Notices*, 81, 2, and details of the motions of the stars have been published in the Greenwich volume on Double Stars. The hypothetical parallaxes for 31 of the remaining 34 systems are given in Table II. Three of the 5 doubtful stars are

also included in this table. The first column gives the number in Burnham's General Catalogue of Double Stars; the second and third columns give the spectral types and photometric magnitudes according to the Henry Draper Catalogue *; the fourth column gives the absolute magnitude corresponding to a parallax of 0″·1, and the fifth column the hypothetical parallax computed with the formulæ given in *Monthly Notices*, 81, 5–10. The sixth to ninth columns give the data from which the hypothetical parallaxes have been computed, the position angles referring to the equinox of date.

The following table shows the distribution of the double stars as to separation near the epoch 1850:—

Separation	0″–1″	1″–2″	2″–4″	4″–8″	8″–16″	16″–32″
Total	147	69	51	57	86	15
Binaries in motion	96	24	4	8	3	0
Optical systems	0	0	1	1	9	4
Uncertain motion	0	0	0	1	4	0

This table shows that 65 per cent. of the stars with separation under 1″ and 35 per cent. of those with separation 1″–2″ have already shown orbital motion, compared with 78 per cent. and 50 per cent. found for the Σ stars from observations extending about 15 years longer.

This result is just what might have been expected, but it should be noted that a number of the closest OΣ stars show as yet no relative motion of the components. Attention might be drawn to the following 14 which, according to Hussey's measures, have a separation not exceeding 0″·5. Now a double star of mass twice that of the sun and a semi-major axis of 100 astronomical units (say separation 0″·5 and parallax 0″·005) will have a period of about 700 years, and a mean angular motion of 30° in 60 years. We should therefore expect these systems to show relative motion unless they are extremely distant. It may be that several of these stars are in fairly rapid motion, but at present it is difficult to distinguish real orbital motion from the errors of observation. These stars should be regularly observed with the most powerful refractors, for they will probably soon prove to be in motion. In any case, they must all be giant stars although all the principal spectral types are represented.

OΣ	Spec.	Mags.	Sep.	OΣ	Spec.	Mags.	Sep.
			″				″
6	A0	7·2, 8·2	0·38	271	F0	8·0, 11·5	0·38
15†	A2, K	7·5, 8·5	0·22	306	G5	7·2, 8·7	0·36
74	F0	8·0, 8·5	0·36	349	G5	7·5, 8·0	0·29
130	B5	6·8, 8·2	0·40	424	...	7·5, 8·7	0·46
150	A0	7·1, 8·0	0·22	487†	A3	6·8, 8·5	0·29
250	F0	7·7, 8·0	0·50	495	F	7·3, 7·5	0·33
267	F5	8·0, 8·0	0·30	510	A	7·5, 7·8	0·40

* For stars between 19^h and 24^h of R.A. the data have been kindly supplied from the manuscripts at Harvard by Dr. Shapley and Miss Cannon.

† There are no reliable observations of OΣ 15 and 487 before 1890.

On the first examination the nature of the relative motion of 5 stars was classified as uncertain. These stars which should all be observed for parallax are:—

OΣ 230. Mags. 7·7, 11·2. Separation, 8″·4—If we assume that this is a binary system we deduce a hypothetical parallax of 0″·076 which gives in conjunction with the Harvard data, Spec. F5, Mag. 8·4, an absolute magnitude of 7·8. As this is improbable for a star of type F5 we conclude that this is an optical double star.

OΣ 274. Mags. 7·0, 10·0. Separation, 14″·6–13″·8. Harvard Spec. Ko. Mag. 7·07. If this is a binary the hypothetical parallax is 0″·095 and the absolute magnitude 7·0. It is probably a binary.

OΣ 323. Mags. 7·4, 10·5. Separation, 7″·2–8″·5. Harvard Spec. Ko. Mag. 8·08. If a binary the hypothetical parallax is 0″·096. Absolute Mag. 8·0. Probably a binary.

OΣ 326. Mags. 7·2, 11·5. Separation, 15″·2–16″·1. If this is a binary the hypothetical parallax is 0″·086, giving an absolute magnitude of 7·8 for a star of spectral type A2. It is probably only an optical double star.

OΣ 440. Mags. 6·2, 10·5. Separation, 12″·4–11″·8. If this is a binary the hypothetical parallax is 0″·069. The spectral type is Ma, and an apparent magnitude of 6·44 gives an absolute magnitude of 5·6. Possibly a binary.

The stars OΣ 274, 323, 440 have been included in Tables I. and II. as binaries, while OΣ 230 and 326 have been included in Table III. as optical systems.

TABLE I.

Binaries showing Relative Motion (*Otto Struve Numbers are given*).

2	53	122	208	256	309	358	416	517
4	62	132	210	260	312	359	418	524
12	65*	139	213	261	313	363	430	527
18	77	149	215	266	315	369	432	531
20	79	156	216	269	323†	375	435	532
28	80	157	218	270	325	377	437	535
31	81	159	220	272	327	382	440†	536*
34	82	170	224	274†	331	383	445	539
38	86	177	227	277	337	387	448	545
42	92	179	228	278	338	392	464	547
43	93	182	229	285	339	395	483	
45	95	185	234	287	341*	400	484	
49	98	187	235	288	351	406	489	
50	112	196	237	296	353	407	500	
51	119	199	241	298	354	410	507	
52	121	201	249	303	357	413	515	

* No hypothetical parallaxes have been computed for the three stars OΣ 65, 341, 536, as the orbits appear to be nearly perpendicular to the tangent plane of projection.

† The three stars OΣ 274, 323, and 440 may be only optical systems.

TABLE II.

This table is additional to those given in *M.N.*, 81, 22–30, and gives the hypothetical parallaxes, etc., in the same form.

B.G.C.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.	Position Angle. 1850.	Position Angle. 1910.	Separation. 1850.	Separation. 1910	
				″	°	°	″	″	
652	Fo	7·20	1·7	·008	321·0	307·1	0·77	0·77	
804	Ko	6·59	3·0	·020	83·5	78·8	4·07	4·07	
892	Ao	7·23	1·7	·008	111·4	143·2	0·46	0·46	
1299	A2	6·51	0·6	·007	111·2	127·4	0·37	0·11	
1378	Fo	7·8	2·9	·011	297·9	290·0	1·51	1·51	
1526	Ao	6·94	2·9	·015	69·2	58·0	1·77	1·77	
1568	F8	7·68	4·7	·025	231·5	198·6	0·92	1·52	
1614	A2	6·35	1·4	·010	150·0	111·7	0·52	0·52	
1849	F5	8·8	4·3	·012	13·	131·	0·30	0·30	
3062	F8	7·34	1·2	·006	191·7	168·0	0·41	0·41	
3074	Ao*p*	2·71	1·4	·055	368·4	338·8	2·00	2·66	N
3353	A3	7·7	1·9	·007	308·0	329·0	0·54	0·54	
4191	A3	7·6	1·9	·007	140·3	118·5	0·54	0·54	
4226	G5	3·68	1·8	·043	231·7	237·1	6·13	6·73	N
4866	A5	3·12	3·2	·102	353·6	360·0	10·53	8·19	N
5023	F2	6·12	4·1	·040	114·0	122·5	5·67	5·67	
5706	K5	7·18	6·1	·061	293·3	302·0	7·98	7·32	N
6155	G5	7·71	2·0	·007	319·3	294·5	0·48	0·48	
6630	F5	4·51	4·0	·079	348·2	356·5	9·44	8·22	N
6671	F2	7·38	3·1	·014	22·6	13·0	1·80	1·80	
6724*	Ko	7·07	7·0	·095*	71·4	64·5	14·59	13·75	
7873*	Ko	8·08	8·0	·096*	108·6	92·5	7·17	8·53	
7900	Fo	6·83	3·3	·019	202·2	207·0	1·62	1·08	
8578	Ao*p*	4·24	−2·1	·005	68·4	49·3	0·46	0·46	
8828	Fo	6·94	−0·3	·004	16·7	27·3	0·39	0·39	
9134	F8	7·00	0·4	·005	43·3	35·0	0·76	0·76	
9535	B8	6·74	−1·1	·003	356·0	348·3	0·42	0·42	
10141	F5	7·02	1·8	·009	131·1	91·9	0·45	0·45	
10590	A3	8·5	7·7	·070	145·0	133·2	6·95	7·65	
10914	Ko	7·62	2·0	·008	197·8	215·7	0·63	0·63	
11011*	M*a*	6·44	5·6	·069*	190·0	184·4	12·35	11·77	
11130	A2	8·75	2·6	·006	107·3	115·2	0·83	0·83	
12088	A2	7·23	1·1	·006	106·	68·	0·30	0·30	
12196	G5	4·56	2·3	·035	356·9	58·5	1·21	0·82	
12573	Ao	6·38	2·0	·013	226·7	270·4	0·62	0·62	

* These three double stars may be only optical combinations.

3074. Positions for 1870 and 1910, not 1850 and 1910.
4226. Abs. trig. par. 0″·024 (2). Spec. par. 0″·028.
4866. Abs. trig. par. 0″·090 (3). Spec. par. 0″·058.
5706. Abs. trig. par. 0″·088 (3). Spec. par. 0″·076.
6630. Abs. trig. par. 0″·060 (2). Spec. par. 0″·050.

Table III.

Optical Systems.

0Σ.	β.G.C.	Mags.	Sep.	0Σ.	β.G.C.	Mags.	Sep.
			″				″
35	884	6·8, 10·0	9·8	393	9786	7·5, 8·4	21·8
104	2668	7·0, 11·0	15·8	411	10473	7·4, 10·2	15·3
154	3562	6·7, 8·4	30·4	425	10675	7·0, 10·5	12·3
165	3844	5·0, 10·7	3·9	460	11472	7·3, 11·7	5·8
230*	5599	7·7, 11·2	8·6	469	11659	7·2, 8·8	31·8
297	7320	7·5, 11·5	13·3	477	11930	7·2, 11·1	9·6
311	7636	7·5, 10·3	13·6	533	10390	4·7, 11·3	10·4
317	7800	7·2, 11·8	15·7	538	7563	5·0, 12·5	21·2
326	7940	7·2, 11·5	15·4				

Hypothetical Parallaxes of some Aitken and Hussey Double Stars.

By J. Jackson, M.A., B.Sc.

(*Communicated by the Astronomer Royal.*)

During the years 1915 to 1922 Dr. Aitken has published seven notes in the *Publications of the Astronomical Society of the Pacific* giving details of the relative motion of some double stars discovered by him. The interval covered by the observations is, with a few exceptions, between 6 and 18 years. Naturally relative motion of the components in such a short interval is only shown by very close binary stars or optical double stars. The hypothetical parallaxes given below have been computed in nearly all cases by the formula

$$\text{hypothetical parallax} = 0\cdot397\ \sqrt[3]{\rho v^2}$$

(see *M.N.*, 81, 10), where ρ is the mean separation and v the annual relative velocity in seconds of arc. For A88 and A417 the orbit elements have been used. From the results deduced it appears that A14 and A1621 may be optical double stars, and that A1518 and A2210 are probably such. These stars should be deleted in any discussion of the results.

The results of the remeasurement of some Hussey stars have been given by Mr. Van Biesbroeck in *Popular Astronomy*, 29, 278. The interval covered by the observations is about 18 years. The results for these stars are given in the second table. It is probable that Hu 783 and 1124 are optical systems, but the large parallaxes of Hu 575, 617, 770, and 1128 appear to be real. This may be tested for the first three of these stars in a few years' time by reobservation, which should show marked curvature. For Hu 1128 there is a proper motion of 0″·75 annually.

For Hu 1176 the hypothetical parallax has been computed from Aitken's elements (*Publ. Astr. Soc. Pacific*, 35, 70).

I am again indebted to Dr. Shapley and Miss Cannon for unpublished material of the Henry Draper Catalogue.

* 0Σ 230 may be a binary.

A.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.
				"
1	F2	8·01	4·1	·017
[14	K2	9·2	8·8	·084]*
53	Ko	8·6	8·3	·087
75	F5	7·54	3·2	·014
88	F8	6·47	3·6	·026
93	G5	8·9	5·1	·017
111 AB.C.	G5	8·9	6·9	·044 ·035
146	F5	7·02	4·2	·028
207	...	9·0	4·1	·010
217	F5	8·5	2·1	·005
218	F5	8·0	2·3	·007
222	F8	8·1	3·0	·009
235	Ko	8·06	4·0	·015
313	F8	8·3	3·8	·013
337	F2	7·75	3·6	·015
347	F2	7·8	3·7	·015
349	...	8·0	4·4	·018
351	...	8·9	6·5	·032
417	Fo	5·56	2·4	·023
431	G5	8·3	4·9	·021
494	F5	5·98	2·8	·023
513	G5	8·91	5·3	·019
519	G5	8·6	5·7	·026
556	Go	7·12	5·8	·055
570	A2	5·90	2·7	·023
578	...	8·1	3·1	·010
606	G5	8·7	4·9	·017
632	Ko	7·8	5·1	·029
655	G5	7·87	3·3	·012
693	F5	8·52	4·1	·013
750	...	8·0	3·6	·013
751	F2 A2	6·82	2·5	·014
758 AB	...	8·8	3·9	·010
764	G5	8·1	4·4	·018
772	F2	8·6	3·8	·011
793	...	7·9	2·2	·007
883	A0	7·31	4·7	·030
889	Ko	9·3	4·9	·013
893	G5	8·7	4·0	·012
948	A5	8·6	2·8	·007
961	F5	8·5	4·9	·019
980	B8	6·48	2·0	·013
1014	F8	8·0	4·7	·022
1034	G5	8·0	3·6	·013
1092	...	8·8	4·5	·013
1095	Go	8·4	3·7	·012
1102	F2	8·19	3·6	·012
1109	F8	7·5	4·8	·028
1152	...	8·7	4·7	·016
1223†	Go	8·5	3·7	·011
1229	Ko	9·6	7·7	·041
1238	F5	7·51	3·9	·019
1256	B9	6·64	0·1	·005
1281	G5	9·2	6·3	·027
1293	Go	7·9	4·1	·017
1303	F5	8·0	3·3	·012
1315	...	8·8	3·8	·010
1404	B9	7·12	1·2	·007
1493	A0	8·6	3·6	·010
1516	F8	7·30	3·4	·016
[1518	G	8·92	9·8	·147]*
1535	G5	8·4	5·5	·027
1544 AB.C.	Fo	8·7	3·8 (4·3)	·011 (·017)
1581	...	9·0	5·3	·019
1584	Go	7·35	4·0.	·022
1609 AB.C.	Ko	8·17	5·0 (6·7)	·023 (·051)
1614	G5	8·7	5·4	·022
[1621	F8	8·8	7·7	·060]*
1627	Fo	7·92	2·8	·010
1653	A3	7·67	2·6	·010
1658	F5	7·64	3·7	·016
1688	A5	8·6	3·6	·010

* A 14, 1518, 1621 are probably optical double stars and should be deleted.
† A 1223, the star C is an optical member of the system.

TABLE—*continued.*

A.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.	A.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.
				″					″
1729	A2	7·18	2·7	·013	[2210	G5	7·37	7·9	·125]‡
1737	A2	8·9	4·2	·012	2244	F8	8·0	4·1	·016
1745	...	9·0	5·4	·020	2329	K5	8·6	6·0	·030
1746 *	A3	7·10	1·9	·009	2414	...	9·2	5·3	·016
AB.C.	G	7·64	2·4	(·032)	2462	F8	8·7	5·5	·023
1755	...	8·5	6·5	·040	2475	A2	9·0	3·9	·010
1765	F5	7·9	2·6	·009	2552	F5	7·7	3·6	·015
1844	F5	6·86	4·2	·029	2579	Fo	8·06	3·7	·013
1851	Ko	9·6	7·2	·034	2586	F8	9·3	5·5	·018
1866	A2*p*	4·86	2·0	·026	2592	F5	7·06	3·1	·016
1903	G5	8·67	4·8	·017	2621	Ao	7·6	2·3	·009
1910	Ao	6·75	1·4	·009	2639	F5	8·8	5·1	·018
1913	...	8·9	3·7	·009	2657	Fo	8·1	3·4	·011
1968	Go	9·0	4·6	·013	2667	A5	6·25	2·5	·018
1971	G5	8·4	5·5	·027	2695	Go	6·35	1·8	·012
1984	Ao	9·21	5·1	·015	2696	F8	8·7	5·3	·021
2003	Go	8·76	4·5	·014	2700	Go	8·9	5·5	·021
2013	K2	9·0	6·1	·027	2703	F5	7·9	3·0	·010
2030	Go	8·6	5·2	·021	2711	Fo	8·3	3·0	·009
2069	F8	7·7	3·5	·015	2757	Ko	8·7	4·4	·014
2080	F2	7·7	2·5	·009	2801	Go	7·50	2·5	·010
2095	F5	8·10	3·2	·010	2855	Ko	8·2	4·8	·021
2100	Fo	6·87	3·1	·017	2889	Ao	7·8	3·0	·011
2131 †	Go	6·67	3·7	·025	2909	Go	7·55	3·6	·016
2163	...	9·0	4·5	·013	2983	G5	8·5	6·0	·032
2167	Go	8·5	4·4	·015	2992	...	8·6	4·6	·015
2172	Ko	8·1	2·3	·007	2993	A2	8·9	4·5	·013
2173	...	8·9	4·4	·013					

* A 1746 is the fainter component of Σ 1224 for which the hypothetical parallax of o″·032 has been determined.

† A 2131, the star C is an optical member of the system.

‡ A 2210 is probably an optical double star and should be deleted.

Hu.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.	Hu.	Spec. Type.	Mag.	Abs. Mag.	Hyp. Par.
				″					″
197	Go	8·17	5·1	·024	608	...	7·8	3·2	·012
198	M*d*	6·5–9·0	0·6–3·1	·007	609	F5	8·0	2·8	·009
280	A5	7·76	1·3	·005	617	...	8·8	8·3	·082
290	G5	9·3	2·5	·004	628	...	8·9	6·4	·032
295	A3	5·30	0·9	·013	767	A5	6·20	1·5	·011
302	...	8·8	2·6	·006	770	...	8·7	9·2	·128
304	A2	5·06	−0·1	·009	[783	B9	8·0	9·0	·162]*
314	Ao	7·9	1·9	·006	803	...	7·9	3·6	·013
371	A3	6·13	0·4	·007	985	G5	9·0	4·7	·014
395	...	8·6	4·3	·013	987	...	7·9	4·7	·023
398	G5	8·4	3·6	·011	992	K2	9·5	8·8	·071
409	...	8·4	4·0	·013	994	B3	6·19	−0·1	·006
413	G5	8·2	4·2	·016	1018	...	8·8	5·4	·017
425	...	8·9	4·9	·016	1080	F8	6·58	5·0	·047
445	Go	8·0	3·6	·013	[1124	Fo	8·22	8·3	·102]*
481	F8	7·8	4·4	·021	1128	Ko	5·48	6·3	·147
494	F5	8·6	3·7	·010	1320	F5	7·7	3·5	·015
546	Go	8·4	5·6	·027	1176	A5	5·38	1·9	·020
575	...	8·5	7·4	·061					

The Masses of Visual Binary Stars of Different Spectral Types.

By J. Jackson, M.A., B.Sc.

(*Communicated by the Astronomer Royal.*)

The masses of stars are determined from their gravitational attraction, and this can only be detected when the separation of the attracting masses does not exceed a few hundred astronomical units, *i.e.* when the separation of the components does not exceed the parallax more than a few hundred times. For spectroscopic double stars the actual attraction is large, but even when the spectra of the two components can be measured, the quantity found from the observations is $m \sin^3 i$, where i is the indeterminable inclination of the orbit plane to the tangent plane. By the application of statistical methods it is possible to deduce the mean mass of a number of systems.

In the case of visual double stars, with which this paper deals, the great difficulty is the determination of the parallax. It is only for a few of the nearer systems that this has been done with the required accuracy and, as is well known, the mass does not differ very much in any case from about twice (say 0·5 to 4 times) the mass of the sun. The

* Hu 783 and 1124 are probably optical systems and should be deleted. In Van Biesbroeck's table Hu 290 and 783 have the numbers interchanged.

data, however, are necessarily selected. Orbits have been determined with more or less accuracy for upwards of a hundred systems. These systems must either be exceptionally massive, or the linear separation must be relatively small. There can be no doubt that the latter provides the greater part of the explanation.

If we assume a mass twice that of the sun we find that 39 hypothetical parallaxes exceed 0″·05. Of these 32 have been substantially confirmed within the errors of observation by the trigonometric method, and in most cases also by the spectroscopic method. For 6 of the remaining stars no parallax determinations have been made, and, in fact, 4 of these are south of declination $-34°$. Two of these stars (β 416 and p Eridani) have hypothetical parallaxes exceeding 0″·1, and might be recommended to anyone taking up parallax work in the southern hemisphere. There remains only one star for which the mass might considerably exceed twice that of the sun. This is ζ Sagittarii, for which the hypothetical parallax is 0″·059, as compared with the trigonometric parallax 0″·018 ± ″·008 found at the McCormick Observatory. This is a star of spectral type A2 and a hypothetical absolute magnitude of 1·6. Its proper motion according to Boss is only 2″·1 per century. An absolute parallax of 0″·023 indicates a mass 33 times that of the sun, although, if we take the parallax as 0″·031, the mass is only 14 times that of the sun. This at once draws attention to the A type stars, and it is found that there are only three in the list for which trigonometric parallaxes are available. These are Sirius, Castor, and the faint double companion to 40 Eridani. For these three systems the trigonometric parallaxes indicate masses of 3·3, 3·3, and 0·4 times that of the sun. As the last is usually regarded as exceptional, the evidence seems to indicate that the A type stars are more than twice as massive as the sun. Krueger 60, of type M*b*, has a mass approximately equal to that of 40 Eridani, and these are probably the two least massive visual binaries for which orbits have been computed.

Considerable confirmation of the reliability of the accepted masses of double stars is obtained from the 4 stars which have been observed both as visual and as spectroscopic binaries. For these stars the linear dimensions of the orbit are deduced independently of direct parallax determination. The data concerning the 4 stars for which the masses are known with relatively great accuracy, are :—

	Spec. Type.	Parallax.	Mass.	Abs. Mag.
		″		
δ Equulei	F5	·067	2·0	2·6
ε Hydræ	F8	·025	3·3	1·6
α Centauri	Go, K5	·76	2·3	4·7, 6·1
Capella	Go	·0632	7·5	−0·8

In order to test the masses independently of the parallax determinations I have attempted to use the proper motions. The data relative to the orbital motion of the stars are given in four previous papers (*M.N.*, **81**, 2, and **83**, 4, 436, 441). When orbit elements were available the only assumption was that the mass was twice that of the sun. For other double stars the statistical considerations involved are explained

in the first of these papers. The proper motions were taken from *Boss's Catalogue*, the *Greenwich Catalogue for 1910*, the *Greenwich Astrographic Catalogue*, vol. iv., and *Publications of the Cincinnati Observatory*, No. 18, Part 4. The proper motions were at the outset divided by the hypothetical parallaxes and the resulting linear motions analysed by Airy's method. In forming the normal equations, each star was included separately and no star was rejected. The following table gives the result of the analysis, the data used in the two parts of the table being quite independent.

Stars with Orbit Elements.

Mean Type.	No. of Stars.	Solar Apex. R.A.	Solar Apex. Dec.	Speed in Astr. Units per Year.	Mean Mass.	Median Hypothetical Parallax.
		°	°			″
A2	21	258·5	+32·0	2·18	13·4	0·019
F5	52	267·5	+29·1	3·07	4·81	·025
G2	30	290·6	+41·6	4·79	1·27	·052
K2	12	269·3	+10·2	8·17	0·26	·056

Stars with Short Arcs only.

		°	°			″
B6	24	278	+32	2·07	12·6	0·012
A2	83	267·2	+35·8	2·16	11·02	·015
F4	102	261·0	+24·2	3·91	1·87	·020
G3	75	293·6	+41·7	6·63	0·38	·024
K1	42	269·3	+25·6	4·87	0·96	·030

The mean mass has been deduced on the assumption that the velocity of the solar system is 19·5 kilometres per second, or 4·12 astronomical units a year. The number of stars in the different groups is not large, but the excessive influence introduced into the ordinary discussions by a few stars of large proper motion is here largely minimised by the change from arc to linear motion. In types B, A, F, all the stars appear to enter into the result, and the exclusion of a few stars would not seriously affect the result. In types G and K, however, the peculiar motions of the individual stars are important and have considerable influence on the result. It is very remarkable that the G type stars in both parts of the table indicate almost the same position of the apex: over 20° different from that generally accepted. In type G the stars may be considered as all dwarfs. In type K there are a number of giants.

For stars with orbit elements the errors in the apparent semi-major axes and the periods are relatively small. For nearly all these stars well determined proper motions are available. The only possible source of error is the adoption of the speed of the solar system relative to these stars, and the general agreement of the direction of this motion with that found from extensive discussions of proper motions seems to indicate that the error cannot be very large. The result may also be influenced to some extent by variations of mass amongst the different stars, but we may conclude that the deduced mass is a close approximation to the actual mean mass of the stars considered.

For the stars for which small areas only of the orbit are available there are several possible sources of error. (1) In the first place, the constant introduced from statistical reasoning cannot be strictly applicable to the stars actually considered. Take, for example, the formula $p = \alpha \sqrt[3]{\rho v^2}$, where p is the hypothetical parallax, ρ the separation, and v the annual relative velocity of the components of a double star. α is a constant introduced from a statistical discussion of the stars as a whole, but clearly our lists will contain those stars for which v is on the whole larger than the average, and exclude those for which it is smaller. In deducing the hypothetical parallaxes we should therefore use a value of α smaller than the average.

(2) Proper motions are available for about half of the stars only. So far as the Boss and Greenwich results are used this should not produce any serious influence. The Cincinnati results, however, were obtained in a systematic search for large proper motions, and so the proper motions used will be on the whole larger than the average, and the resulting mass unduly small. The following table shows the sources of proper motion :—

Type.	Boss.	Greenwich.	Cincinnati.
B	19	5	0
A	63	19	1
F	47	43	12
G	26	21	28
K	14	13	15

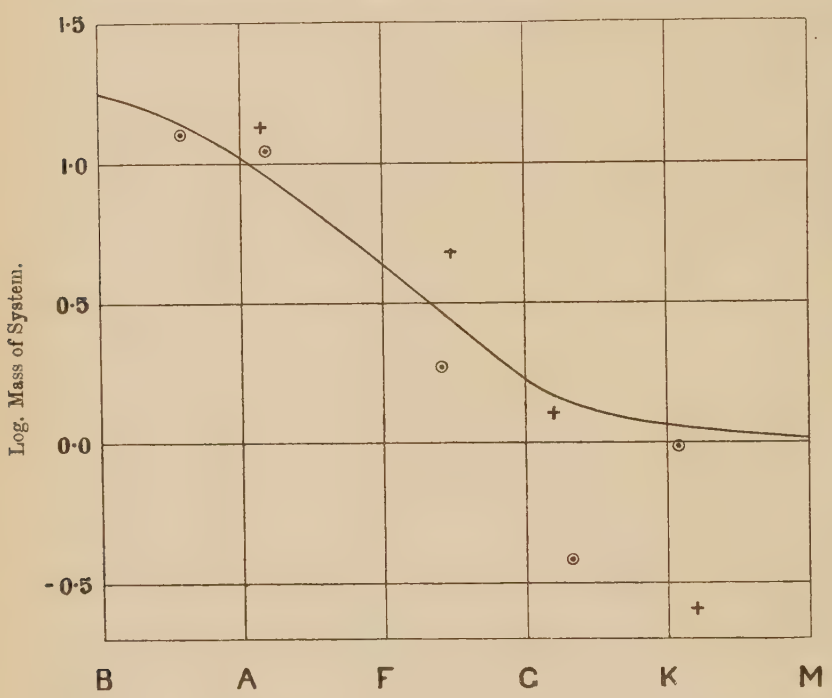
(3) Errors of observation will also influence the results when the observed relative motion of the components is small. This will be specially the case with spectral types B and A, which include a considerable proportion of stars for which orbital motion can just be detected. Accidental errors may therefore have led to the deduced hypothetical parallaxes and masses being too large.

Of these three sources of error (1) and (3) tend to make the resulting masses, particularly those of early type stars, too large, while (2) tends to make the stars of late type too small, and possibly to a very considerable extent.

The results obtained in the above discussion are shown on the following diagram, which also shows the curve obtained by Seares * from a discussion of the absolute magnitudes. It may be considered that there is some slight evidence that the slowly revolving stars are on the whole less massive than those for which the relative motion has been large enough for the computation of orbit elements. Nevertheless the difference is not great. Considering the difference of treatment the agreement with Seares's results is close. In Seares's discussion the hypothetical parallaxes from small arcs were reduced to agree with those from orbit elements. This assumes equality of masses and ignores the basis on which the parallaxes were computed. In fact it only really utilises the data available for stars with orbit elements. Reasons for this treatment were given in systematic differences found in (*a*) the absolute magni-

* *Astrophysical Journal*, 55, 165, or *Mount Wilson Contribution*, No. 226.

tudes, and (*b*) the parallaxes. In (*a*) a comparison was made for different spectral types, and as far as the data went it indicated a correction of the magnitude applied. But in (*b*) no reference was made to spectral type, and as the hypothetical parallaxes are based on the same mass for all types the different types ought to be considered separately, especially when the differences between the hypothetical and spectroscopic results are to be used for the determination of masses. The argument, based on a few stars for which I have computed a hypothetical parallax both



+ Stars with Orbit Elements. ⊙ Stars with short arcs.

The Curve is from Seares' discussion.

from an arc and orbit elements, is also of little consequence, as in all these cases the orbit elements are of little value. However, the correction applied by Seares is in the same direction to that indicated by (1) and (3) above. On the whole, the results indicate smaller masses for late type stars than those deduced by Seares. There are very few data for stars earlier than B8 or later than K2.

It should be noted that Seares's results are geometrical mean masses, while those obtained here may be considered as the cubes of arithmetical means of cube roots of the masses. If the variation in mass is not large the difference involved will not be very great.

The results obtained seem to indicate that the average masses of binary systems falls from about 12 times the mass of the sun for late B type stars to equality with the sun for K type. It is generally assumed that the variations of mass within a spectral class is not large. Yet when we turn to special stars for which the masses are well determined we fail to find the large masses indicated in general discussions. There are, for example, two A0 type stars with well-determined masses. These are Sirius and 40 Eridani, with masses 3·3 and 0·4 times the mass of the sun. For Castor, also of type A0, the orbit elements are not very good, but the parallax is fairly well known. The mass is practically equal to that of Sirius. It is not improbable that the very large mean masses obtained for stars of early type may be the result of selection.

In this paper some attention may be drawn to the fact that several very close double stars observed by Otto Struve show as yet no certain relative motion. The following list gives, for example, a few stars for which the mean angular motion for 50 years has been computed on the assumption that the absolute magnitude is $-1\cdot0$, and the mass the same as that of the sun.

0Σ.	Spec.	Separation.	Angular Motion in 50 years.
		"	°
6	A0	0·38	9
15	A2, K8	0·22	22
130	B5	0·40	10
150	A0	0·22	10
349	G5	0·29	10
487	A	0·29	20
495	F	0·33	18
510	A	0·40	23

Clearly, unless these stars begin to show relative motion soon they must be extremely brilliant or have very small mass.

On Jeans's Theory of the Origin of Spiral Nebulæ.
By Harold Jeffreys, M.A., D.Sc.

The purpose of this note is to attract attention to certain internal inconsistencies in Jeans's theory of the origin of spiral nebulæ. The predictions of this theory are so strikingly in accordance with the results of modern observations of the forms and motions of lenticular and spiral nebulæ, as to afford good evidence for the truth of this theory or some closely similar one; but the account given of the method of transition from the lenticular to the spiral form appears to be dynamically incorrect.

Jeans's argument * is that if a mass is rotating with angular velocity

* *Problems of Cosmogony and Stellar Dynamics*, Cambridge University Press, 1919, p. 139.

ω about the axis of z, and if p and ρ are the pressure and density at the point (x, y, z) of the mass, the equations of motion are

$$\left.\begin{aligned}\frac{\partial p}{\partial x} &= \rho\frac{\partial V}{\partial x} + \omega^2\rho x\\ \frac{\partial p}{\partial y} &= \rho\frac{\partial V}{\partial y} + \omega^2\rho y\\ \frac{\partial p}{\partial z} &= \rho\frac{\partial V}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

in which V is the potential of the whole gravitational field of force, including tidal forces if any are present. Thus we may write

$$V = V_M + V_T \, . \quad . \quad . \quad . \quad . \quad (2)$$

where V_M is the gravitational potential of the rotating mass under consideration, and V_T is the potential of the tidal field.

Now it will be seen that these equations are all incomplete. The correct equations of motion of a perfect fluid are

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial V}{\partial x} \quad . \quad . \quad . \quad . \quad (3)$$

with two similar equations; where u, v, and w are the velocity components, and $\frac{d}{dt}$ indicates differentiation following a particle of the fluid.

Jeans has written

$$u = -\omega y, \quad v = \omega x, \quad w = 0 \, . \quad . \quad . \quad . \quad (4)$$

making

$$\frac{du}{dt} = -\omega^2 x; \quad \frac{dv}{dt} = -\omega^2 y; \quad \frac{dw}{dt} = 0 \, . \quad . \quad . \quad (5)$$

But it will be seen that (4) will be true only if the motion is one of pure rotation; if any particle has an outward motion or one parallel to the axis, or if its velocity is not uniform, the accelerations will deviate from the values (5), and the deviations should appear in the equations (1). When the nebula is moving under its own gravitation the motion may be one of pure rotation, and equations (1) will hold as they stand; but if tidal forces are present, the outer boundary of the mass will in almost all cases cease to be an exact surface of revolution, and hence every particle moving around it must have some inward and outward motion at various parts of its path. Thus equations (1) cannot be accurate when tidal forces are present. On the other hand, the tidal accelerations may be expected to be small. This is probably true in the case of a nebula; but in all known problems of tidally deformed fluid masses the tidal accelerations have been found to be comparable with the accelerations that would be produced if the particles were under the influence of no forces but the tidal ones, and therefore the errors committed are probably comparable with the terms depending on V_T, which have been retained. Thus so long as the phenomena considered are such that tidal forces do not affect them to any important extent, we should expect the

equations (1) to give a sufficient approximation, either with or without the tidal terms; but if a stage is reached when tidal forces control the character of the evolution, it will be necessary to include the small acceleration terms in these equations.

Jeans then writes

$$\Omega = V + \tfrac{1}{2}\omega^2(x^2+y^2) \quad . \quad . \quad . \quad . \quad (6)$$

and proves that Ω is a constant over the boundary of the fluid. This enables him to prove that the mass, if in pure rotation, will become lenticular in form, a sharp edge developing around the equator when the condensation has proceeded far enough. So far the theory is in close accordance with observations of lenticular nebulæ.

At this stage Jeans begins to take into account the small tidal forces due to neighbouring stars and other nebulæ, for which purpose his result (6) is inadequate, for the reason already given. He shows (*loc. cit.*, p. 213) that the most important part of the tidal potential will be a second harmonic, so that we can write

$$V_T = S_2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and then

$$\Omega = V_M + S_2 + \tfrac{1}{2}\omega^2(x^2+y^2) \quad . \quad . \quad . \quad (8)$$

He goes on: "When the external potential S_2 is omitted, the condition for a break-up ($\partial\Omega/\partial x = 0$) becomes

$$-\frac{\partial V_M}{\partial r} = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the break-up commences simultaneously at all points of the cross-section in the plane of xy. But when the term in S_2 is included in Ω the cross-section in the plane of xy can no longer be circular; it becomes slightly elliptical, and obviously the break-up will occur first at the two ends of the major axis of this ellipse. Thus instead of a ring of matter being thrown off, we see that matter will be thrown off initially only at two antipodal points."

Even if we admit for purposes of argument that Ω is constant over the bounding surface, this argument breaks down at two stages. The condition for either a sharp edge or a point on the boundary is that

$$\frac{\partial\Omega}{\partial n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where differentiation takes place along the outward normal. If this condition holds at all points of the equatorial section of the mass, the equator will be a sharp edge; if it holds only at a finite number of points, these will be conical points, and the remainder of the equatorial section will consist of ordinary points.

In the present problem, if

$$V_M = \frac{\mu}{r} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$S_2 = A(x^2 - y^2) \quad . \quad . \quad . \quad . \quad (12)$$

where μ and r are constants, the form of the equatorial section is given by

$$\frac{\mu}{(x^2+y^2)^{\frac{1}{2}}}+\tfrac{1}{2}\omega^2(x^2+y^2)+\mathrm{A}(x^2-y^2)=\text{const.} \quad . \quad . \quad (13)$$

If we write

$$x^2+y^2=\varpi^2 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$y=x\tan\phi \quad . \quad . \quad . \quad . \quad (15)$$

this becomes

$$\frac{\mu}{\varpi}+\tfrac{1}{2}\omega^2\varpi^2+\mathrm{A}\varpi^2\cos 2\phi=\text{const.} \quad . \quad . \quad . \quad (16)$$

The condition for a point or a sharp edge is

$$-\frac{\mu}{\varpi^2}+\omega^2\varpi+2\mathrm{A}\varpi\cos 2\phi=0 \quad . \quad . \quad . \quad (17)$$

For a sharp edge around the equator these must be consistent for all values of ϕ; which is impossible, since that would imply that ϖ was constant: in other words, that the equator was a perfect circle and that A was zero, which is not the case. Hence the equator must develop conical points; the ellipse is not an approximation to its form. Now in the widely distended nebula $-\mu\varpi^{-2}+\omega^2\varpi$ is negative; hence as the nebula contracts $-\mu\varpi^{-2}+\omega^2\varpi+2\mathrm{A}\varpi\cos 2\phi$ will change sign first for those points where $\cos 2\phi=+1$. Hence two conical points will be formed, one immediately below, and one directly opposite to the disturbing body, and in the further development the ejected matter will pour out from these. Thus ejection to form arms is possible if Ω is constant over the boundary, but not for Jeans's reasons.

The fact that Ω is not constant over the boundary will complicate this argument, and possibly destroy its validity. For revolving matter to pass a sharp point on its boundary would require an infinite acceleration locally, which is impossible. A complete theory would require the inclusion of the tidal acceleration terms, and ejection would commence when the matter at the boundary has an acceleration towards the centre less than $\omega^2\varpi$. The condition is therefore that

$$-\frac{\mu}{\varpi^2}+\omega^2\varpi+2\mathrm{A}\varpi\cos 2\phi=0 \quad . \quad . \quad . \quad (18)$$

at the actual point of ejection; but to ascertain whether this is ever possible in a dynamical system would require a further investigation. If the points of high tide are unaffected by rotation, it will presumably be possible; but they may be inverted, and then ejection may take place either at the two points 90° distant from them, as seen from the centre, or not at all.

In conclusion, the forms of lenticular nebulæ agree with the forms predicted by Jeans's theory as far as equation (6) of this paper, which suggests that the theory is valid for them. The spiral nebulæ, to judge from the continuous gradation that appears to exist between the lenticular and spiral forms, are probably a later stage in the evolution of the lenticular nebulæ, starting from the stage when the lenticular nebulæ develop

sharp edges. Jeans's suggestion that the nature of the transition is fundamentally affected by small tidal forces is plausible, but the theory of the transition remains to be constructed.

Note added June 27.—I have been granted the opportunity of seeing in proof Dr. Jeans's reply to this criticism. In this he no longer supposes Ω constant over the boundary, nor that the condition for a break-up is $\partial\Omega/\partial\varpi = 0$, thus withdrawing the essential point that I challenge. His new proof quite meets my objections, and it is perhaps a sufficient justification of this note that it has called forth his further elucidation of an important physical point.

On the Tidal Distortion of Rotating Nebulæ. By J. H. Jeans, Sec. R.S.

1. In the foregoing note Dr. Jeffreys raises doubts as to the accuracy of my view of the events to be expected in a rotating nebula which has reached the critical lenticular state and is acted on by a small external tidal field of force. The apparent difference between Dr. Jeffreys and myself arises, I think, largely from our different points of view, Dr. Jeffreys regarding the problem as a mathematical one, and I as a physical one. The following mathematical investigation is, however, believed to be rigorous, and, as will be seen, in so far as Dr. Jeffreys differs from me, it confirms my view as against his.

2. As usual, we suppose the mass to rotate about the axis Oz in the direction from Ox to Oy. Let the time of a complete rotation be $2\pi/\omega$, and let the system be referred to axes rotating in space with the constant angular velocity ω. The co-ordinates of any point of the mass relative to the moving axes being x, y, z, we denote its velocity relative to these moving axes by $\dot{x}, \dot{y}, \dot{z}$, and its actual velocity in space by u, v, w. Then

$$u = \dot{x} - \omega y, \quad v = \dot{y} + \omega x, \quad w = \dot{z} \quad . \quad . \quad . \quad (1)$$

The components of hydrodynamical rotation ξ, η, ζ are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad . \quad . \quad (2)$$

or, in terms of the relative velocities, $\dot{x}, \dot{y}, \dot{z}$, by

$$\xi = \frac{\partial \dot{z}}{\partial y} - \frac{\partial \dot{y}}{\partial z}, \quad \eta = \frac{\partial \dot{x}}{\partial z} - \frac{\partial \dot{z}}{\partial x}, \quad \zeta = \frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + 2\omega \quad . \quad . \quad (3)$$

The accelerations in space are given by *

$$f_x = \frac{\partial u}{\partial t} - \omega v + \dot{x}\frac{\partial u}{\partial x} + \dot{y}\frac{\partial u}{\partial y} + \dot{z}\frac{\partial u}{\partial z}$$

$$f_y = \frac{\partial v}{\partial t} + \omega u + \dot{x}\frac{\partial v}{\partial x} + \dot{y}\frac{\partial v}{\partial y} + \dot{z}\frac{\partial v}{\partial z}$$

$$f_z = \frac{\partial w}{\partial t} \qquad + \dot{x}\frac{\partial w}{\partial x} + \dot{y}\frac{\partial w}{\partial y} + \dot{z}\frac{\partial w}{\partial z}.$$

* Lamb, *Hydrodynamics* (4th ed.), p. 12.

Using the rotations ξ, η, ζ as given by (2), these may be expressed in the form

$$f_x = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \omega^2(x^2+y^2)] + (\dot{z}\eta - \dot{y}\zeta), \text{ etc.} \quad . \quad (4)$$

The equations of motion are

$$f_x = \frac{\partial V}{\partial x} - \frac{\partial}{\partial x}\int\frac{dp}{\rho} \quad . \quad . \quad . \quad . \quad (5)$$

so that $\partial f_x/\partial y = \partial f_y/\partial x$, etc., whence, from (4),

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + (\dot{z}\eta - \dot{y}\zeta)\right] = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + (\dot{x}\zeta - \dot{z}\xi)\right].$$

Thus we may put

$$\frac{\partial u}{\partial t} + \dot{z}\eta - \dot{y}\zeta = \frac{\partial \Psi}{\partial x}, \text{ etc.} \quad . \quad . \quad . \quad . \quad (6)$$

where Ψ is a function of x, y, z and t not yet determined. The components of acceleration are now

$$f_x = \frac{\partial \Psi}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \omega^2(x^2+y^2)]$$

and the equations of motion (5) have the common integral

$$V + \tfrac{1}{2}\omega^2(x^2+y^2) - \int\frac{dp}{\rho} = \Psi + \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad . \quad . \quad (7)$$

3. We now put $V = V_M + V_T$ where V_M is the potential arising from the mass itself and V_T is the external potential of the tidal field, which we assume to be small. When $V_T = 0$ there is a solution corresponding to steady rotation, in which $\dot{x}$, $\dot{y}$, $\dot{z}$ all vanish, so that Ψ reduces to a constant and equation (7) reduces to

$$V_M + \tfrac{1}{2}\omega^2(x^2+y^2) - \int\frac{dp}{\rho} = \text{constant} \quad . \quad . \quad (8)$$

The boundary is of course given by $\int\frac{dp}{\rho} = 0$, or

$$V_M + \tfrac{1}{2}\omega^2(x^2+y^2) = \text{constant} \quad . \quad . \quad . \quad (9)$$

When V_T is not zero, but a small quantity of the first order, there will be a solution in which all quantities differ from their values in the foregoing solution by small quantities of the first order. In particular, $\dot{x}$, $\dot{y}$, $\dot{z}$ will be small quantities of the first order, so that $\dot{x}^2$, $\dot{y}^2$, $\dot{z}^2$ may be neglected, and equation (7) reduces to

$$V + \tfrac{1}{2}\omega^2(x^2+y^2) - \int\frac{dp}{\rho} = \Psi \quad . \quad . \quad . \quad (10)$$

But Ψ now differs from a constant by a small quantity of the first order, and the boundary is given by

$$V + \tfrac{1}{2}\omega^2(x^2+y^2) = \Psi \quad . \quad . \quad . \quad . \quad (11)$$

4. The case of special interest is that in which V_T is proportional to a second-degree harmonic, say

$$V_T = A(x^2 - y^2),$$

referred to fixed axes. If $\varpi^2 = x^2 + y^2$, and θ is an azimuth, this may be written

$$V_T = A\varpi^2 \cos 2\theta, \text{ referred to fixed axes,}$$

or

$$V_T = A\varpi^2 \cos 2(\theta - \omega t), \text{ referred to rotating axes} \quad . \quad (12)$$

Corresponding to any equilibrium solution (ϖ, z, θ) under no tidal forces, we shall now show that when small tidal forces are present of the type specified in equation (12), there is a steady-motion solution $(\varpi + \Delta\varpi, z + \Delta z, \theta + \Delta\theta)$ in which, neglecting small quantities of the second order,

$$\Delta\varpi = a \cos 2(\theta - \omega t), \quad \Delta z = b \cos 2(\theta - \omega t), \quad \Delta\theta = c \sin 2(\theta - \omega t) \quad (13)$$

in which a, b, and c are functions of ϖ and z only.

It is at once seen that the change of density produced by the displacements specified in (13) depends on θ and t only through the factor $\cos 2(\theta - \omega t)$. Thus the gravitational potential of the distorted mass will be of the form

$$V_M = B + C \cos 2(\theta - \omega t),$$

where B and C are functions of ϖ and z only, B being of course the value of the potential in the undistorted state.

Turning back to equation (10), it is now clear that each term on the left-hand side will be of the general form

$$\alpha + \beta \cos 2(\theta - \omega t),$$

where α and β are functions of z and ϖ only. Thus Ψ must also be of the same form, say

$$\Psi = \lambda + \mu \cos 2(\theta - \omega t) \quad . \quad . \quad . \quad . \quad (14)$$

where λ and μ are functions of z and ϖ only.

The internal velocities $\dot{\varpi}$, $\dot{z}$, $\dot{\theta}$ corresponding to the small displacement (13) are

$$\dot{\varpi} = 2\omega a \sin 2(\theta - \omega t), \quad \dot{z} = 2\omega b \sin 2(\theta - \omega t), \quad \dot{\theta} = -2\omega c \cos 2(\theta - \omega t) \quad (15)$$

Treating the internal velocities as small quantities of the first order, equations (6) assume the form

$$\frac{\partial\Psi}{\partial x} = \frac{d\dot{x}}{dt} - 2\omega\dot{y} \quad . \quad . \quad . \quad . \quad (16)$$

$$\frac{\partial\Psi}{\partial y} = \frac{d\dot{y}}{dt} + 2\omega\dot{x} \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{\partial\Psi}{\partial z} = \frac{\partial\dot{z}}{dt} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and these are equivalent to the three equations of motion.

Transforming to cylindrical co-ordinates (ϖ, z, θ) by the substitution

$$x = \varpi \cos\theta, \quad y = \varpi \sin\theta,$$

these three equations transform into

$$\begin{aligned}\frac{\partial\Psi}{\partial\varpi} &= \frac{1}{\varpi}\left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right)\\ &= \frac{d}{dt}\dot{\varpi} - 2\omega\varpi\dot{\theta}\\ &= -4\omega^2 a\cos 2(\theta-\omega t) + 4\omega^2\varpi c\cos 2(\theta-\omega t) \quad . \quad . \quad (19)\\ \frac{\partial\Psi}{\partial z} &= \frac{d}{dt}\dot{z} = -4\omega^2 b\cos 2(\theta-\omega t) \quad . \quad . \quad . \quad . \quad . \quad (20)\\ \frac{\partial\Psi}{\partial\theta} &= x\frac{\partial\Psi}{\partial y} - y\frac{\partial\Psi}{\partial x}\\ &= 2\omega\varpi\dot{\varpi} + \varpi^2\frac{d}{dt}\dot{\theta}\\ &= -4\omega^2\varpi a\sin 2(\theta-\omega t) - 4\omega^2\varpi^2 c\sin 2(\theta-\omega t) \quad . \quad . \quad (21)\end{aligned}$$

On substituting the value for Ψ given in equation (14), it appears that these three equations are satisfied if

$$\begin{aligned}&\frac{\partial\lambda}{\partial\varpi} + \frac{\partial\mu}{\partial\varpi}\cos 2(\theta-\omega t) = -4\omega^2(a-\omega c)\cos 2(\theta-\omega t)\\ &\frac{\partial\lambda}{\partial z} + \frac{\partial\mu}{\partial z}\cos 2(\theta-\omega t) = -4\omega^2 b\cos 2(\theta-\omega t)\\ &-2\mu\sin 2(\theta-\omega t) = 4\omega^2(\varpi a + \varpi^2 c)\sin 2(\theta-\omega t).\end{aligned}$$

From a consideration of the solution in the steady state it is clear that

$$\frac{\partial\lambda}{\partial\varpi} = 0, \quad \frac{\partial\lambda}{\partial z} = 0,$$

so that all the equations are satisfied if

$$\mu = -2\omega^2(\varpi a + \varpi^2 c) \quad . \quad . \quad . \quad . \quad (22)$$

$$\frac{\partial\mu}{\partial\varpi} = -4\omega^2(a - \varpi c) \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{\partial\mu}{\partial z} = -4\omega^2 b \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

These equations depend only on z and ϖ. Since μ is a linear function of a, b, c, they are in effect three linear equations for the determination of a, b, and c in terms of A. Without solving these equations in detail, it has been shown that a tidal field of the kind we have had under consideration will result in all cross-sections of the rotating mass perpendicular to its axis of rotation assuming an elliptical shape. The major axes of these ellipses all lie in the same direction, and this

direction is either that of the tidal pull, or is at right angles to this direction. Which of these two directions is the true one depends upon the sign of a/A. I see no simple means of determining this sign by direct calculation, but it is easily seen by a physical argument that high tide must lie in the direction of maximum tidal pull. For in the special case in which $\omega = 0$ the equilibrium configuration is spherical, and here it is well known that when distortion takes place high tide occurs in the direction of tidal pull. For this state of things to change as ω^2 increases there must be some configuration for which a/A changes sign, and since infinite values of a/A are known to be out of the question, the configuration must be one for which $a = 0$. Now, on putting $a = 0$ in equations (22) and (23) we find either that $c = 0$ or that μ varies as $1/\varpi^2$. The latter alternative can be dismissed, since it would make μ infinite along the axis of rotation, so that we must have $c = 0$ and hence also $\mu = 0$ and $b = 0$; in other words, the tidal field produces no distortion of any kind. This being entirely out of the question, we conclude that a/A cannot change sign for any value of ω, and hence that for all rotating masses high tide is in the direction of maximum tidal pull.

The harmonic we have considered, namely, $V_T = A(x^2 - y^2)$, is of course not the most general harmonic of the second degree. The most general harmonic of the second degree can be made up of harmonics of this general type together with harmonics of the types

$$z^2 - \tfrac{1}{2}(x^2 + y^2) \text{ and } xz.$$

The tidal forces arising from harmonics of the first of these types will result only in the rotating mass undergoing a slight distortion while retaining its circular cross-section; the tidal forces arising from harmonics of the second type merely set up a slight rotation about an axis in the plane of the nebula. From the evolutionary point of view the harmonics we have already discussed are the only ones which possess any importance.

6. The condition that a break-up shall begin at any point of the surface is that

$$\frac{\partial}{\partial n}\left(\int \frac{dp}{\rho}\right) = 0.$$

For a point on the equator this assumes the form (*cf.* equations (10) and (19))

$$\frac{\partial V}{\partial \varpi} + \omega^2 \varpi = \frac{\partial \Psi}{\partial \varpi} = 4\omega^2(\varpi c - a)\cos 2(\theta - \omega t),$$

and on substituting the values for ϖ and $\partial V/\partial \varpi$ on the equator we obviously get an equation of the type

$$p + q \cos 2(\theta - \omega t) = 0.$$

As the rotation increases this equation will first be satisfied when $\cos 2(\theta - \omega t)$ is equal to either $+1$ or -1, so that break-up will first occur at the ends of either the major axis or of the minor axis of the elliptic cross-section. There is now no difficulty in discriminating

between the two possibilities; every single agency tending to break-up is more potent at the end of the major axis than at the end of the minor axis, so that we may conclude that break-up will first occur at the two ends of the major axis. Thus there will be a continuous stream of particles shot off from the two ends of the major axis. At these points equations (15) show that $\dot{\varpi}=0$, $\dot{z}=0$, while $\dot{\theta}$ does not vanish. Thus the particles start their careers with purely tangential velocities, but these velocities differ by first-order terms both from the velocity appropriate to a circular orbit and from the velocity determined by the rotation of the nebula.

7. To sum up, mathematical analysis appears to have confirmed every statement made in my sentence challenged by Dr. Jeffreys in the paper which precedes this one. By way of summary I may perhaps rewrite the sentence in such a way as to incorporate the further knowledge that has been gained as follows:—

When the external tidal potential S_2 is absent, break-up commences simultaneously at all points of the cross-section in the equatorial plane *xy*. But when the term S_2 is included, the cross-section in the plane of *xy* can no longer be circular; it becomes slightly elliptical, and it has been proved that break-up will occur first at the two ends of the major axis of this ellipse. Thus instead of a ring of matter being thrown off, we have seen that matter will be thrown off initially only at two antipodal points; we have further seen that these points are those two points on the equator of the rotating mass at which the tidal pull is greatest.

Deduction of the Moon's Parallactic Inequality and Allied Terms from the Greenwich Lunar Observations 1900 *to* 1922. By A. C. D. Crommelin, D.Sc., and R. T. Cullen.

(*Communicated by the Astronomer Royal.*)

This paper may be looked on as a continuation of those of Dr. Cowell on the derivation of the solar parallax from the Greenwich Observations of the Moon, 1847 to 1901, *M.N.*, **64**, 85, 579, 694. It deals with the period 1900·0 to 1923·0, 1½ years thus overlapping with Cowell's period. The first part of the paper includes all the observations both of the Moon's limb and the crater Mösting A made both with the Transit Circle and the Altazimuth. The second part is confined to the Mösting A observations, beginning 1905·0, with the two instruments, the question of the Moon's semidiameter being thus eliminated.

1. The errors of the Moon's tabular longitudes dealt with in this first portion of the paper are based on Hansen's tables with Newcomb's corrections, without any further corrections, and are derived from:—

(1) Transit Circle observations of the limbs.

(2) Transit Circle observations of the crater Mösting A beginning 1905.

(3) New Altazimuth observations of limbs (and Mösting A from 1905).

(4) New Altazimuth extrameridian observations of the limbs.

2. The errors of longitude were arranged according to the age of the Moon in the form Tabular – Observed (the mean error for each year being taken out) and further subdivided into months. Owing to the scarcity of data near new Moon the investigation was confined to observations made 10 days or less before and after full Moon. The grouping of the observations had been carried out before the revision of all the Greenwich lunar observations was decided on, which is the reason why the various corrections applied in that revision were not introduced here.

The means are shown in the following table; the numbers of observations on which these means depend range from 17 downwards, with an average of about 10. The cases where the number is less than 5 are indicated.

TABLE I.

Tabular Errors of the Moon's Longitude arranged according to the Age of the Moon and each Month of the Year. (Mean Error for each Year taken out.)

Age of Moon in Days.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
4	+1″·0	−0″·4	−1″·7	−1″·5	−2″·2	−1″·7	−2″·0	−2″·4	−1″·4	−0″·3	0″·0	+0″·2
5	+0·5	−0·9	−2·5	−2·9	−2·8	−2·7	−2·9	−1·3	−0·3	−1·2	−0·4	−0·2
6	−0·1	−1·8	−2·7	−2·2	−2·6	−2·9	−1·8	−0·7	−0·9	−1·1	0·0	−0·5
7	−0·3	−1·5	−1·7	−2·0	−2·8	−3·1	−2·4	−1·1	−1·7	−0·7	+0·7	−0·4
8	−0·9	−2·2	−2·3	−2·5	−2·9	−3·0	−2·2	−2·0	−0·8	−0·5	−0·2	+0·1
9	−1·3	−2·6	−2·9	−3·2	−2·7	−2·0	−1·4	−1·7	−1·1	−0·3	−0·3	−0·5
10	−1·7	−3·1	−2·2	−2·9	−2·8	−2·3	−0·2	−1·9	−1·2	−0·5	+0·1	−0·6
11	−2·1	−2·6	−2·8	−2·3	−2·3	−1·4	−1·3	−1·4	−1·1	−0·1	+0·4	−0·9
12	−2·3	−2·3	−1·9	−1·8	−0·7	−0·4	−0·9	+0·1	+0·2	0·0	0·0	−0·9
13	−1·3	−1·6	−1·7	−1·3	−0·9	−0·4	+0·5	+0·9	+1·2	0·0	+0·1	−0·9
14	−0·3	+0·2	−0·3	0·0	+0·1	−0·6	+1·5	+1·7	+1·7	+2·0	+0·6	−0·2
15	+0·5	0·0	−0·1	+0·1	+1·5	+1·4	+3·0	+3·0	+2·4	+2·7	+1·5	+1·0
16	+0·6	+0·2	+0·8	+1·2	+1·5	+2·5	+2·6	+3·3	+3·0	+1·8	+1·2	+0·4
17	+1·0	+0·9	+2·0	+2·1	+1·9	+3·1	+4·9	+3·5	+2·9	+1·9	+1·6	+1·8
18	+1·0	0·0	+2·0	+1·7	+4·6	+4·4	+3·7	+3·9	+3·2	+2·6	+1·4	+0·1$_3$
19	+1·1	+1·1	+1·8	+1·2	+3·1	+4·2	+4·1	+4·8	+1·9	+1·1	+1·6	+1·7
20	+1·2	+0·6	+2·3	+2·2	+3·4	+3·6	+3·9	+2·5	+1·7	+1·0	+0·9	+1·8
21	+2·0	+1·3	+2·6$_3$	+1·7$_3$	+3·7	+3·1	+4·4	+2·3	+1·3	+0·2	+1·1	+1·6
22	+0·9	+1·3	+1·4$_2$	+1·5	+2·3$_4$	+2·6	+2·8	+2·1	−0·1	+0·6$_4$	+0·9	+0·7
23	−0·6	...	+0·7$_3$	+1·3$_3$	+3·6	+2·2	+1·1	+0·7	+1·2	−0·4	+0·7	+0·9

3. The quantities in Table I. were treated as being of equal weight, and analysed for each month by the formula

$$a + b \sin D + c \cos D + d \sin 2D + e \cos 2D,$$

D being the mean elongation of the Moon from the Sun. The results for each month are :—

TABLE II.

Jan.	−″·38	−0″·51 sin D	+ ″·14 cos D	+1″·13 sin 2D	−″·74 cos 2D
Feb.	− ·46	−1·27	+ ·57	+ ·78	− ·12
Mar.	− ·73	−2·15	− ·34	+ ·62	− ·67
Apr.	− ·46	−2·12	+ ·11	+ ·64	− ·16
May	+ ·13	−3·16	+ ·07	+ ·42	− ·24
June	− ·15	−3·03	− ·89	+ ·69	− ·59
July	+ ·33	−2·86	−1·15	+ ·74	− ·36
Aug.	+ ·37	−2·16	−1·03	+1·17	− ·14
Sept.	+ ·53	−1·31	− ·44	+1·01	+ ·49
Oct.	+ ·28	−0·76	− ·58	+ ·76	+ ·39
Nov.	+ ·43	−0·59	− ·32	+ ·30	− ·13
Dec.	+ ·10	−0·76	− ·10	+ ·38	− ·52

The mean values for the twelve months are

$$″·00 − 1″·72 \sin D − ″·33 \cos D + ″·72 \sin 2D − ″·23 \cos 2D.$$

Further analysis led to the expressions

$$″·00 − ″·48 \sin g' − ″·19 \cos g'$$

for the first term,

$$+″·76 \sin (g' + D) + ″·56 \cos (g' + D) − ″·41 \sin (g' − D) − ″·08 \cos (g' − D)$$

for the D terms, and

$$− ″·20 \sin (g + 2D) − ″·11 \sin (g' − 2D) − ″·07 \cos (g − 2D)$$

for the 2D terms, where g' is the Sun's mean anomaly.

The term $−″·48 \sin g'$ is in good agreement with Cowell's value $−″·55 \sin g'$. The term $−″·23 \cos 2D$ is in good agreement with the theoretical correction that should be applied, because in the mean, the constant of +10″ (the mean tabular error of longitude for the period 1900–22) has been applied to the residuals, and this involves applying $+0″·2 \cos 2D$ to the tabular values (see *M.N.*, **64**, 585).

The term $+″·76 \sin (g' + D)$ is close to the value $+″·67 \sin (g' + D)$ found by Cowell (*M.N.*, **65**, 136). The term $+″·56 \cos (g' + D)$ is probably partly due to the term $−″·68 \sin (g + 2\varpi − 3V − 5E)$ omitted by Hansen, but introduced into the Mösting A investigation which follows. The two terms have nearly equal periods. The second argument is 71° in excess at 1900·0, and 16° in arrear at 1923·0.

Cowell finds no correction to Hansen for the term in $\sin (g' − D)$.

4. Observations of Mösting A, simultaneously with the first or second limb, give regular determinations of the Moon's semidiameter applicable to the observations. This increases the weight of the determination of the parallactic term very largely.

In Dr. Cowell's investigation where this was not possible the weight was very largely reduced on this account.

(1) Comparisons between Moon's limb and Mösting A with the Transit Circle from 1906 to 1922 indicated a correction to Hansen's R.A. semidiameter of $-0''\cdot22$ from Transit Circle observations.

(2) Similar comparisons with the Altazimuth from 1905 to 1915, a correction of $+0''\cdot22$ from Altazimuth observations.

The Transit Circle series extends over a longer period, and contains somewhat more observations in the years common to both. A weight of 3 to 1 was given to (1) and (2), and a mean correction of $-0''\cdot11$ adopted.

If this correction be applied to the semidiameter the coefficient of the correction to the parallactic term will be reduced from $-1''\cdot72$ to $-1''\cdot61$, a figure in close agreement with $-1\cdot70$ obtained later from the observations of Mösting A. The value $-1\cdot61$ gives a solar parallax of $8''\cdot784$.

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5. The analysis of errors up to this point has been based on the original tabular – observed longitudes. In the following analysis these quantities have been corrected by the formulæ (partly due to Cowell) that are printed in *M.N.*, 83, 360.

Observations of the lunar crater Mösting A have been carried on both with the transit-circle and the altazimuth (in the meridian) since the beginning of 1905.

The observations in North Polar Distance from 1905 to 1910 were discussed in *M.N.*, 71, 526, for the deduction of the parallax of the Moon, being combined with the observations made at the Cape of Good Hope.

In Dr. Cowell's research on the parallactic inequality as given by the observations of the Moon's limb with the transit-instrument and transit-circle (period 1847 to 1901), 5647 observations were available, but he noted that the difficulty of separating the error of semidiameter from that of the coefficient of sin D reduced the weight to that of 1030 observations. As the crater had been observed on 1075 days up to the end of 1922, and as a determination of the parallactic inequality from the crater is free from the uncertainty arising from the semidiameter, though the presence of shadow in the crater when D is near 90° or 270° introduces a somewhat analogous difficulty, it was thought worth while to analyse the observations for this purpose. The annual means of the residuals were then applied with reversed signs. Table I. gives the residuals grouped by 5° of D for the periods 1905 to 1914, and 1915 to 1922. The former period contains observations with both instruments, the latter with the transit-circle only.

TABLE III.

Limits of D.	1905–1915. Mean of Residuals.	1905–1915. No. of Obs.	1915–1922. Mean of Residuals.	1915–1922. No. of Obs.	Combined Residuals.
° °	″		″		″
95–100	+ ·50	10	− ·57	4	+ ·19
100–105	+ ·54	18	− ·26	14	+ ·19
105–110	+ ·91	18	− ·25	18	+ ·33
110–115	+ ·13	20	− ·18	20	− ·02
115–120	+ ·29	17	− ·41	16	− ·05
120–125	− ·53	29	− ·68	24	− ·60
125–130	− ·12	20	− ·03	10	− ·09
130–135	− ·39	16	−1·01	15	− ·69
135–140	− ·62	26	−1·16	16	− ·83
140–145	−1·02	21	−1·74	16	−1·33
145–150	− ·52	26	− ·98	20	− ·72
150–155	− ·23	27	− ·35	14	− ·27
155–160	− ·42	20	− ·48	15	− ·45
160–165	− ·43	18	− ·04	18	− ·24
165–170	− ·16	25	− ·59	15	− ·32
170–175	− ·27	17	−1·27	12	− ·69
175–180	+ ·14	21	− ·52	12	− ·12
180–185	+ ·41	16	− ·35	12	+ ·09
185–190	− ·47	18	− ·61	17	− ·54
190–195	+ ·35	24	− ·33	9	+ ·16
195–200	·00	15	− ·51	12	− ·23
200–205	+ ·88	19	+1·07	11	+ ·95
205–210	− ·23	18	+ ·60	12	+ ·10
210–215	+1·16	17	+ ·34	8	+ ·90
215–220	+ ·71	12	+ ·75	14	+ ·73
220–225	+ ·74	18	+ ·30	7	+ ·62
225–230	+1·51	13	+ ·64	13	+1·07
230–235	+ ·82	14	+1·98	9	+1·27
235–240	+1·08	20	+1·24	13	+1·14
240–245	+ ·53	19	− ·19	8	+ ·32
245–250	+ ·50	11	+ ·58	10	+ ·54
250–255	+ ·54	16	+ ·58	5	+ ·55
255–260	+ ·52	8	+1·38	9	+ ·98
260–265	+1·44	15	− ·08	9	+ ·87
265–270	+ ·40	7	+ ·81	7	+ ·60
270–275	+3·37	2	...	...	+3·37

Analysing these with weights proportioned to the number of observations, we obtain the following representation of the residuals.

1905–1915	+″·11 sin D +″·70 sin 2D
1915–1922	−″·20 sin D +″·61 sin 2D
1905–1922	−″·14 sin D +″·58 sin 2D

A solution was also obtained in which the first 6 and last 6 lines were given diminished weight, on account of the presence of shadow in the crater: the result was −″·14 sin D +″·59 sin 2D, so that the change in the weighting had no appreciable effect on the formula.

The following are the residuals left after applying +″·14 sin D −″·58 sin 2D +″·20 cos 2D to the above residuals:—

TABLE IV.

D.	Residuals.	D.	Residuals.	D.	Residuals.	D.	Residuals.
97½°	+″·29	142½°	−″·63	187½°	−″·51	232½°	+″·55
102½	+ ·39	147½	− ·03	192½	+ ·07	237½	+ ·41
107½	+ ·63	152½	+ ·39	197½	− ·44	242½	− ·40
112½	+ ·38	157½	+ ·15	202½	+ 63	247½	− ·14
117½	+ ·43	162½	+ ·30	207½	− ·33	252½	− ·08
122½	− ·04	167½	+ ·14	212½	+ ·38	257½	+ ·42
127½	+ ·53	172½	− ·33	217½	+ ·14	262½	+ ·39
132½	− ·03	177½	+ ·14	222½	− ·04	267½	+ ·21
137½	− ·14	182½	+ ·23	227½	+ ·37	272½	+3·08

The second term, taken alone, indicates that Hansen's co-efficient of the variation 2370″·14 should be diminished by ″·58, making it 2369″·56 (agreeing closely with the investigation earlier), Brown's value being 2369″·91; the value now found differs considerably from Cowell's value, obtained from the observations in the period 1847–1901, which was 2370″·21.

Hansen's theoretical value was 2369″·75, in sensible agreement with Delaunay and Hill.

It may be noted that since we have in the mean applied + 10″ to the original residuals, we should also have applied + 1″·1 cos g + ″·2 cos 2D + ″·2 cos (2D − g) (see *M.N.*, **64**, 585); the second term alone affects the residuals. It has no sensible effect on the coefficients of sin D, sin 2D, but its application does produce an appreciable improvement in the final residuals, after the terms − ″·14 sin D + ″·58 sin 2D have been taken out.

Using the formula in *M.N.*, **64**, 585, and noting that the tabular value for the parallactic coefficient has been already increased by 1″·56, we find for the observed value of the coefficient − 126″·46 + 1″·56 + 0″·14 = − 124″·76.

Comparing this with Brown's value − 125″·154 for parallax 8″·80549, we find 8″·778 for the solar parallax. Cowell's value was 8″·787, with an estimated probable error of ″·018. It will be seen that the two results differ by only half of the probable error of the latter.

The results were also analysed by separate months in order to test the dependence of the coefficients on the sun's mean anomaly.

The following corrections to the tabular values were thus obtained:—

$$\sin D\{-″·18 + ″·53 \cos (g' - 2°·8)\}$$
$$+\sin 2D\{+″·51 + ″·33 \sin (g' + 40°·3)\}$$

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6. Another investigation was made with the Mösting A observations, to examine what value they indicated for the annual term in the Moon's Physical Libration. The most direct way of doing this is to group the differences, Mösting A—Limb, by months.

The results are as follows:—in solutions (1) the months are weighted equally, in (2) proportionally to the number of observations.

	Transit Circle.	Altazimuth.
(1)	$+^{s}·022 \sin (g' + 20°·9)$	$+^{s}·038 \sin (g' + 38°·7)$
(2)	$+^{s}·024 \sin (g' + 27°·6)$	$+^{s}·040 \sin (g' + 43°·0)$

Giving the Transit Circle double weight, we adopt as the coefficient $^{s}·028 = ″·42$. This is equivalent to $-92″ \sin g'$ for the correction required to the selenocentric libration. The *Berliner Jahrbuch* has applied $-59″ \sin g'$ for 1909 and subsequent years. Up to 1907 $-132″ \sin g'$ was applied, and for 1908 $-48″ \sin g'$.

The value deduced from the Greenwich observations is $-151″ \sin g'$, and it suggests that the value used up to 1907 in the *B.J.* was nearer the truth than that subsequently adopted there.*

* Our attention has been called to Lt.-Col. Stratton's paper (*R.A.S. Mem.*, 59, 257). His final adopted value is $-129″ \sin g'$.

M. Puiseux in the Annual Report of the Paris Observatory for 1915 gives the value $-1200'' \sin g'$, which he deduced from Paris photographs. The Greenwich observations appear to be inconsistent with so large a value.

A second method of deducing this coefficient is by analysis of the monthly means of the residual (tabular – observed) of the longitude of Mösting A. This involves the assumption that the annual equation is completely corrected by the quantity $+''{\cdot}55 \sin g'$ already applied (see *M.N.*, 83, 360). The assumption is justified by the tolerable agreement between the result of this method and that of the preceding one.

The residuals for the separate months are :—

Jan. $+''{\cdot}15$, Feb. $-''{\cdot}05$, Mar. $+''{\cdot}30$, Apr. $+''{\cdot}63$, May $+''{\cdot}64$, June $+''{\cdot}36$
July $-''{\cdot}04$, Aug. $-''{\cdot}12$, Sept. $-''{\cdot}29$, Oct. $-''{\cdot}59$, Nov. $-''{\cdot}42$, Dec. $-''{\cdot}28$

These give the formula $+''{\cdot}51 \sin (g' - 25^\circ{\cdot}6)$, or in selenocentric longitude $-112'' \sin g'$, giving for the total coefficient $-171'' \sin g'$.

It may be noted that no term has been announced in the Physical Libration with a period of one lunation. Librations with any other period would have no appreciable effect on the parallactic inequality deduced from Mösting A.

7. In conclusion, the three different determinations of the solar parallax are collected, together with some other recent determinations.

	Solar Parallax.
Cowell from Limb observations, 1847–1901, applying $-''{\cdot}47$ to semidiameter	$8''{\cdot}787 \pm ''{\cdot}018$
Present investigation, Limb and Mösting A, 1900–1922, applying $-''{\cdot}11$ to semidiameter	8·784
Present investigation, Mösting A only, 1905–1922	8·778
Edzard Noteboom (*Ast. Nach.*, 5122–3), Eros perturbations by Earth	8·799 ± ·001
Constant of aberration, with Cookson Floating Telescope	8·815 ± ·006
Constant of aberration, excluding plates with large residuals	8·810 ± ·004
Newcomb's combined value from aberration	8·781
Newcomb's combined value from Moon's parall. ineq.	8·773
From observations of Eros	8·806 ± ·004
Evershed, radial motion of Venus	8·77

The values obtained for the parallactic inequality from Mösting A and the limbs are in good agreement, and indicate a solar parallax very near $8''{\cdot}78$. The agreement of different investigations seems to give a determination from the parallactic inequality more weight than has usually been assigned to it. It may be remarked that all recent determinations by gravitational methods give a parallax less than $8''{\cdot}80$. The determination by Noteboom, which is very near this value, is entitled to far greater weight than other determinations by gravitational methods.

Royal Observatory, Greenwich :
1923 *June* 7.

The Construction of a Chart giving the Angular Distances of Stars from, and the Position Angles relative to, the Ant-apex of the Solar Motion.
By W. M. Smart, D.Sc.

(*Communicated by Professor A. S. Eddington, F.R.S.*)

1. In the many problems connected with the proper motions of the stars, two quantities are of fundamental importance—the angular distance of a star from the apex or ant-apex of the solar motion, and the position angle of the star relative to the apex or ant-apex. These quantities are obtained either by direct computation or by reference to previous investigations in which the quantities are tabulated for particular stars or by employing special tables constructed usually for a totally different purpose. For example, Kapteyn * has tabulated the angular distances of the Bradley stars for three different positions of the apex. Again, a limited range of the values of the quantities concerned can be obtained, fairly easily, by inspection or by a little manœuvring, from certain navigation tables.† I describe here a graphical method applicable to all stars. Briefly, the angular distances and position angles are read off from a chart on which curves of equal angular distances and equal position angles are drawn in each case at intervals of 5°. From the known R.A. and Declination of a star, it is thus possible to obtain at a glance from the chart, with the requisite accuracy, both the angular distance and the position angle.

2. I have taken the position of the ant-apex to be:—

$$\text{R.A. } 6^h : \text{Declination } -34^\circ.$$

If future researches showed that this value of the R.A. was in error, the usefulness of the chart would not be impaired thereby—an adjustment of the origin of R.A. co-ordinates is all that would be necessary. The value of the Declination is more uncertain.

3. In devising a graphical method one is forced to seek that which yields the most simple geometrical constructions. The gnomonic projection seemed to offer the greatest advantages in this respect. The final curves are, accordingly, based on a set of curves obtained by a gnomonic projection which involves intrinsically a system of familiar curves, the conics. These can be easily and accurately drawn without much labour. The only alternative seemed to be the laborious calculation of the co-ordinates of a large number of points on each curve.

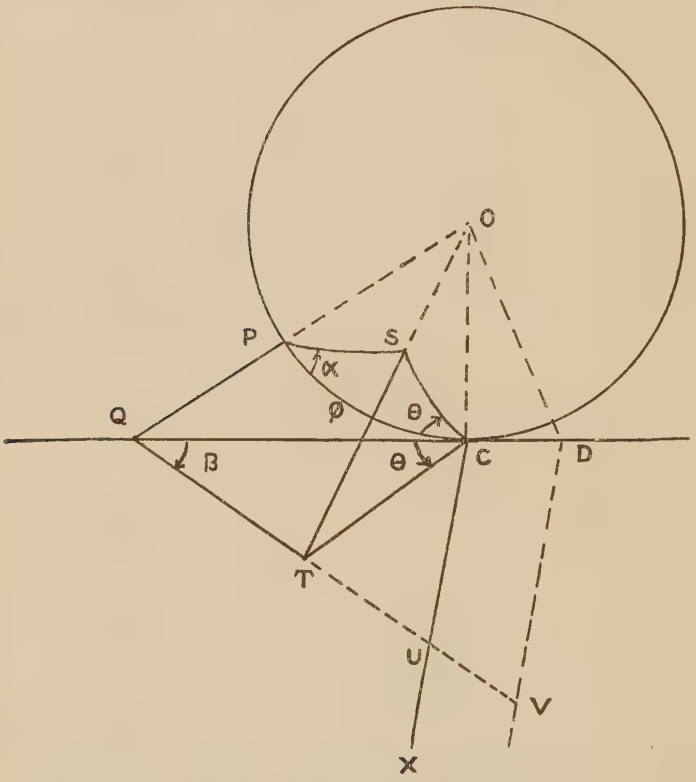
It is evident that the curves between R.A. 0^h and 12^h are symmetrical about R.A. 6^h, and that the curves between 12^h and 24^h are simply the reflection relative to the axis of Right Ascension of the curves between R.A. 0^h and 12^h. Thus it is necessary to construct only the curves between 0^h and 6^h for all declinations.

The centre of projection may be chosen conveniently in several ways—only the two projections actually employed are described below.

4. *Method* (*a*).—C, the centre of projection, is taken to be at the

* *Groningen Publications*, No. 7.
† Percy Davis's Alt-Azimuth Tables: Towson's Tables.

ant-apex and the tangent plane XCY at C is the plane of projection. P is the South Pole. The projection of a point S on the sphere is obtained by joining O (the centre of the sphere) to S and producing this line till it meets the tangent plane at T. All great circles project into straight lines on the tangent plane. Thus the central meridian CP projects into the straight line CQ and this is taken to be the y-axis. CX, perpendicular to CQ, is the x-axis. The meridian PS projects into QT and the great circle CS into CT. If α is the difference of the R.A.'s



of C and S, then α is represented on the chart by β, the angle CQT. Also, the spherical angle SCP is equal to the angle QCT. PC is denoted by ϕ where $\phi = 56°$.

(i) Relation between β and α.

Let x, y denote the co-ordinates of T.

Then

$$x = \mathrm{CT} \sin\theta\,;\ y = \mathrm{CT} \cos\theta.$$

Also if the radius of the sphere is unity, $\mathrm{CT} = \tan \mathrm{CS}$.

Thus

$$x = \tan \mathrm{CS} \sin\theta\,;\ y = \tan \mathrm{CS} \cos\theta.$$

From the spherical triangle PCS, we have

$$\cos\phi\cos\theta = \sin\phi\cot \mathrm{CS} - \sin\theta\cot\alpha$$

or

$$\cos\phi[\tan \mathrm{CS}\cos\theta] = \sin\phi - \cot\alpha[\tan \mathrm{CS}\sin\theta].$$

The equation of QT is thus

$$y\cos\phi + x\cot\alpha = \sin\phi,$$

from which it follows that

$$\tan\beta = \cos\phi\tan\alpha.$$

(ii) Relation between angles PSC and QTC.

We have

$$-\cos \mathrm{PSC} = \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi$$

and

$$\sin \mathrm{PSC}\cos \mathrm{CS} = \cos\alpha\sin\theta + \sin\alpha\cos\theta\cos\phi.$$

By division,

$$-\cot \mathrm{PSC}\sec \mathrm{CS} = \frac{1 - \tan\theta\tan\alpha\cos\phi}{\tan\theta + \tan\alpha\cos\phi}$$

$$= \cot(\theta+\beta) \qquad \text{from the relation in (i).}$$

This gives the desired relation, viz.

$$\tan \mathrm{QTC} = \cos \mathrm{CS}\tan \mathrm{PSC}.$$

(iii) The meridians.

From (i), it is seen that the meridians are straight lines on the chart, radiating from Q and making with the y-axis an angle β given by

$$\tan\beta = \cos\phi\tan\alpha.$$

If QT intersects the x-axis in U, then

$$\mathrm{CU} = \sin\phi\tan\alpha.$$

This relation is employed in drawing the meridians on the chart.

(iv) The parallels of Declination.

We have

$$\cos \mathrm{PS} = \cos \mathrm{PC}\cos \mathrm{CS} + \sin \mathrm{PC}\sin \mathrm{CS}\cos\theta$$

or

$$\cos \mathrm{PS}\sec \mathrm{CS} = \cos\phi + y\sin\phi.$$

Since

$$\sec^2 \mathrm{CS} = 1 + x^2 + y^2$$

we obtain, by squaring and rearranging, the equation of a parallel of Declination on the chart, viz.

$$x^2\sin^2\delta + y^2(\sin^2\delta - \sin^2\phi) - 2y\sin\phi\cos\phi + \sin^2\delta - \cos^2\phi = 0.$$

This represents

(α) an ellipse if $\pm\delta$ lies between 56° and 90°,

(β) a parabola if $\pm\delta = 56°$,

(γ) a hyperbola if $\pm\delta$ lies between 0° and 56°.

The system of ellipses and hyperbolæ are put in the form

$$\frac{x^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

where

$$a^2=-\frac{1+\cos 2\delta}{\cos 2\delta+\cos 68^\circ}$$

$$b=-\frac{\sin 2\delta}{\cos 2\delta+\cos 68^\circ}$$

$$y_0=-\frac{\sin 68^\circ}{\cos 2\delta+\cos 68^\circ}.$$

The curves are easily drawn when the values of a^2, b, y_0 have been computed.

The equation of the equator is easily seen to be

$$y=-\cos\phi.$$

The equator is the line DV in the figure.
The meridian QT cuts DV in V where

$$\mathrm{DV}=\mathrm{cosec}\,\phi\tan\alpha.$$

This result was utilised in checking the accuracy of the plotted meridians.

(v) The angular-distance curves.

For points on the sphere at a constant distance from C, the locus is a circle on the chart with centre at C and of radius tan CS. It is this geometrical simplicity that is the principal advantage of the gnomonic projection. These circles were drawn on the gnomonic chart for every 5° up to 80° in the value of CS. The R.A. and Declination of several points on each circle were then read off. By plotting these positions on squared paper—R.A. horizontal and Declination vertical—on a scale of 10° to 1 inch in both co-ordinates, one obtains the final curves of angular distance in a convenient form.

(vi) The position-angle curves.

The position-angle CSP is given by

$$\tan(\pi-\mathrm{CSP})=\sec\mathrm{CS}\tan(\theta+\beta).$$

It is possible to use this relation fairly effectively in a simple geometrical manner: the work was found somewhat tedious, and this particular method was soon abandoned.

The analytical equation of these curves was also found difficult to employ, except with a vast amount of labour. The equation is derived thus (in polar co-ordinates):—

We have

$$\cos\mathrm{CS}\cos\theta=\sin\mathrm{CS}\cot\phi-\sin\theta\cot\mathrm{CSP}$$

and since $r=\tan\mathrm{CS}$

the equation becomes

$$(r \cot \phi - \cos \theta)^2 = \sin^2 \theta \cot^2 \mathrm{CSP}(1 + r^2)$$

which is evidently too complicated for numerical work.

The simplest method of drawing the position-angle curves was to employ directly the angular-distance curves drawn on the squared paper. From the spherical triangle PCS we have

$$\sin \alpha \sin \phi = \sin \mathrm{CS} \sin \mathrm{CSP}.$$

A small table was constructed from this formula to give α in terms of CS and CSP for 5° intervals of these quantities. Thus the value of α for the point at which a particular curve of the system, CSP = const, cuts the curve, *e.g.* CS = 60°, is read off directly from the table and the point of intersection plotted on the squared paper. A smooth curve drawn through several points similarly obtained is the curve desired. The method evidently depends on the accuracy with which the curves CS = const have been drawn.

5. *Method* (*b*).—If in the figure the point of tangency C is now the pole and P represents now the ant-apex, the parallels of declination are the system of circles with C as centre and radii tan p, where p is the polar distance. The angular-distance curves on the gnomonic chart are the systems of conics already drawn. The meridians are straight lines radiating uniformly from C. The same chart is used in both methods. Where there is a lack of sensitiveness, say in method (*a*), the curves can be used as for method (*b*) usually to give results of adequate accuracy.

6. The final curves have been tested fairly thoroughly by comparison with Kapteyn's values for the Bradley stars and with Davis's Alt-Azimuth tables so far as these are applicable. The angular distances and position angles are generally required to the nearest degree: the chart is amply competent to yield this degree of accuracy except within 10° of the apex or ant-apex. In these regions the necessity for an accuracy of this order disappears.

In conclusion, I wish to express my indebtedness to Miss A. V. Douglas, M.Sc., for valuable and considerable assistance in the preparation of the chart.

Observatory, Cambridge:
1923 *May.*

Note.—The Charts referred to in the above paper are being published separately, on 8 sheets about 12 inches square, and may be obtained by applying to the Assistant Secretary, R.A.S., Burlington House, London, W 1.

Note on the Determination of Absolute Magnitude from λ_e and λ_m.
By Knut Lundmark, Ph.D., and W. J. Luyten, Ph.D.

1. In a previous paper* we have tried to show that the combination of λ_{eff} and λ_{min} is not, as yet, suitable for the practical determination of absolute magnitude. A method, using these two colour equivalents, was proposed by Lindblad as an attempt to find the absolute magnitude of a star without a knowledge of its spectral class.† From the material given by Lindblad, we computed the correlation coefficient r of the relation between λ_e and λ_m to be as high as:

$$r = 0{\cdot}97 \pm 0{\cdot}008 \ (m.e.)$$

thus showing that λ_m can be computed with extreme accuracy from λ_e itself. To avoid misunderstanding, we must again emphasise that we do not question that there is a dependency on absolute magnitude for both λ_e and λ_m. We think, however, that these phenomena are too closely parallel to enable us to compute the absolute magnitude from the difference in their behaviour.

In a subsequent paper,‡ Lindblad has commented on the conclusions of our paper, and states that "λ_m is obviously relatively more sensitive to a change in luminosity within the same spectral class than is λ_e." According to him it is permissible to exclude all stars with $\lambda_e < 4180$ in the computation of r, "because for these stars it would be futile to try a discrimination between different degrees of luminosity by means of λ_m and λ_e." It must be remarked, however, in connection with this, that the range in absolute magnitude for the whiter stars is comparable with that of the yellow stars. In the F stars, *e.g.* the absolute magnitude varies from -3 for the Cepheids to $+5$ for the dwarfs. We do not wish to emphasise the range of twenty magnitudes between Plaskett's star and β Orionis on one hand and O_2 Eridani and Van Maanen's F-star on the other hand. However, it is just for such early type stars that Lindblad has applied his method of estimating the absolute magnitude from the intensity of the cyanogen bands, and in his last note he again connects this procedure with his method involving λ_m.

Out of the 64 stars originally selected on the basis of absolute magnitude, Lindblad, in his note, excludes 19 with $\lambda_m < 4180$ A and recomputes the correlation constant r to be

$$r = 0{\cdot}76 \pm 0{\cdot}06.$$

We notice that among the nineteen excluded stars the absolute magnitudes range from $+6{\cdot}2$ to $-2{\cdot}5$ (not including P Cygni), the whole range for all his stars being from $+11{\cdot}2$ to $-2{\cdot}5$. In spite of the selection in the excluded material, this coefficient seems still too high to allow the treating of λ_m and λ_e as independent quantities in the determination of M.

2. A practical test to which the determination of M from λ_m and λ_e may be put concerns the accuracy of the predicted M. In his note

* *M.N.*, 82, 495, 1922.
† *Upsala Årsskrift*, 1920.
‡ *M.N.*, 83, 97, 1922.

Lindblad claims on the one hand, that the mean accidental errors in his values of λ_m and λ_e may be estimated below 10 A for both wave-lengths. On the other hand, he declares that "a considerable part of the dispersion in λ_e for late type stars (see fig. 2 in our previous paper) can be accounted for by errors in spectral classification and errors of measurement." In our opinion this can only be so when the accidental errors of measurement are much larger than 10 A; the value adopted both by the Greenwich observers and by Lindblad. Between spectral classes K5 and M the effective wave-length reaches a maximum, and accordingly the influence of errors in spectral class reaches a minimum value. The actual range in λ_e is from 4440 to 4560, *i.e.* twelve times the value adopted for the probable error.

The estimate for the accidental error given by Lindblad is derived mainly from errors of setting on the plates and does not give an adequate idea of the external mean error. A mean error of 16 A for these measures has been derived by Lundmark from a comparison of values for stars common to several papers of Lindblad, and may be considered better than an estimate. Even this value is certainly too low, since no account has been taken of the considerable systematic error in the measurement of λ_e mentioned in Lindblad's note. In his previous paper, differences of 30 A were found to exist between measures made at different times (*Årsskrift Upsala*, p. 75).

From the diagram given in Lindblad's note we see that for $\lambda_e =$ 4200 A,

11 stars with $\overline{M} = 0{\cdot}5$, $\overline{\lambda}_m = 3764$ A

and

12 stars with $\overline{M} = 5{\cdot}8$, $\overline{\lambda}_m = 3744$ A.

The difference in λ_m between giants and dwarfs, of the same λ_e, is 20 A, or less than the systematic error mentioned above. It thus appears doubtful whether a combination of λ_m and λ_e can be used, even for discriminating between giants and dwarfs. Further corroboration of this is afforded by the fact that the three stars for which Lindblad predicted absolute magnitudes of the order of 8^M to 9^M have been found by Luyten to have no large proper motion. Probably they have a much higher magnitude than that estimated by Lindblad.

3. The best idea of the relation between λ_e, λ_m, absolute magnitude and spectral class can perhaps be found by representing Lindblad's data by a formula of the form

$$\lambda_e = \lambda_m + a + b\mathrm{M} + c\mathrm{S} \quad . \quad . \quad . \quad . \quad (1)$$

It is only when the constants a, b, c are known, together with their mean errors, that we can discuss quantitatively the possibility of deriving M from λ_e and λ_m.

From a plot of $\lambda_e - \lambda_m$ against spectral class, as shown in fig. 1, it is evident that the spectral class has to be taken into account in the above solution. Instead of the spectral class we might also have used λ_e itself. A plot of λ_e against $\lambda_e - \lambda_m$ is given in fig. 2. Here we have indicated the degree of luminosity by three different symbols.

The line drawn at $\lambda_e = 4180$ leaves to the left of it all stars for which, according to Lindblad, "it would be futile to attempt a discrimination between different degrees of luminosity by means of λ_e and λ_m." In our opinion we should substitute the second line, drawn at $\lambda_e = 4240$, in the above statement, thus excluding in first instance all stars with $\lambda_e < 4240$ or earlier than about K2. Both figs. 1 and 2 show a much more definite relationship than does fig. 3, where we have plotted $\lambda_e - \lambda_m$ against M.

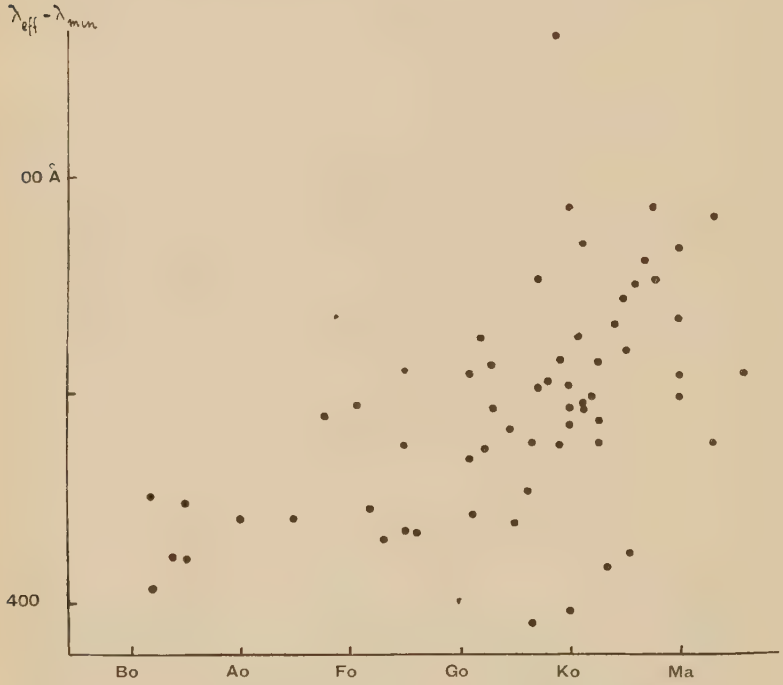


FIG. 1.—Relation between $\lambda_e - \lambda_m$ and Spectral Class.

By taking S = 0·0 for Ao, and S = 4·0 for Ma we find from Lindblad's wave-lengths and the most accurate absolute magnitudes and spectra available:

$$\left.\begin{array}{ll} a = 419{\cdot}3 & \pm 3{\cdot}70\,\text{Å} \\ b = 1{\cdot}38 & \pm 0{\cdot}37 \\ c = 9{\cdot}12 & \pm 1{\cdot}37 \end{array}\right\} \text{from 59 equations of condition.}$$

That this solution has a real meaning may be judged from the fact that the sum of the squares of the residuals is reduced from 176200 A^2 to 30300 A^2. The mean error for unit weight appears to be $\pm 23{\cdot}3$A.

The value of b indicates that for a change in $\overline{M}$ of ten absolute magnitudes there is only a change of 13·8 A in $\lambda_e - \lambda_m$.

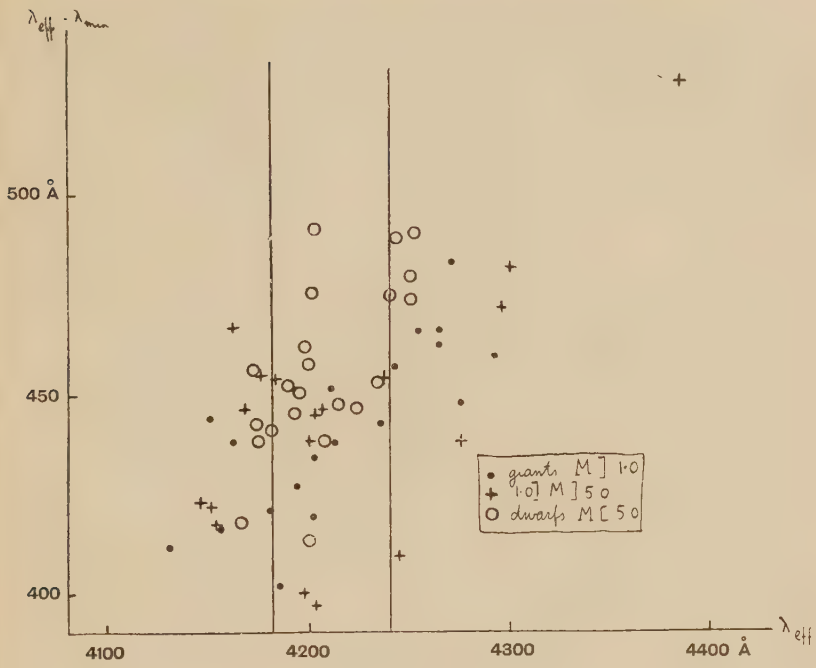


FIG. 2.—Relation between $\lambda_e - \lambda_m$ and λ_e.

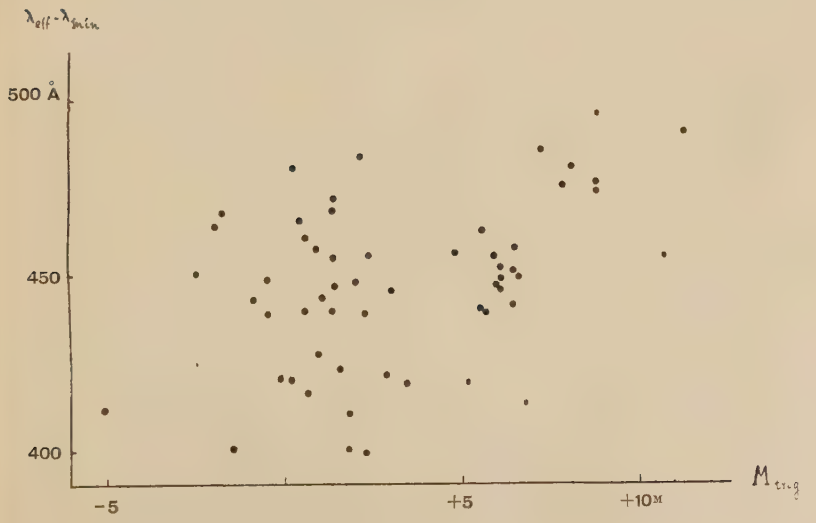


FIG. 3.—Relation between $\lambda_e - \lambda_m$ and Absolute Magnitude.

From the numerical data given above, we conclude that M can be determined in this way with a mean error of ± 3 magnitudes. According to Kapteyn's luminosity law, the distribution of the absolute magnitudes of all stars in space follows an error curve with 7·7 as mean value and ± 2·5 as mean error.

With $c = 0$, or, in other words, attributing all the difference in $\lambda_e - \lambda_m$ to absolute magnitude we find:

$$a = 441{\cdot}2$$
$$b = \quad 2{\cdot}30$$

corresponding to a change of 23 A for 10 M.

As we have only 60 stars at our disposal we have not thought it worth while to consider terms of higher order in our solution.

It may be recalled that a very conservative estimate of the errors in Lindblad's measures was 16 A, giving 22 A for the error in $\lambda_e - \lambda_m$; furthermore, a value of 23 A was found above by solving equation (1).

4. Summarising we wish to emphasise again that we have never questioned the theoretical possibility of a relationship between λ_m, or any other colour-equivalent, and the absolute magnitude. On the other hand, we think we have shown conclusively in the present paper that, as yet, the actual determination of absolute magnitude from λ_e and λ_m alone is subject to such large errors that even a discrimination between giants and dwarfs seems doubtful.

1923 *May* 3.

Stellar Velocity and Absolute Magnitude: note on a Paper by Professor Eddington and Miss Douglas. By W. S. Adams and G. Strömberg.

In an interesting paper in *Monthly Notices* for 1923 January, Professor Eddington and Miss Douglas have called attention to the systematic effect upon the relationship between stellar velocity and absolute magnitude of the accidental errors in the absolute magnitudes as determined spectroscopically. When the proper motions are used for the determination of this relationship it is clear that a correction must be applied. The effect is due to the fact that the same observed absolute magnitudes enter into both of the variables between which the correlation is derived, into the one directly, into the other indirectly, through the conversion of angular motion into linear velocity.

It is evident that this effect will be most pronounced when the dispersion in absolute magnitude among the stars investigated is equal to or smaller than that due to accidental errors in the determinations of absolute magnitude. Obviously in such a case a correction must be applied. In fact, if the dispersion in absolute magnitude is not larger than that due to accidental errors in the determinations, the relationship derived between velocity and absolute magnitude must necessarily be very uncertain, unless the number of stars used is very large. For this reason, in our discussion of this relationship in Mount Wilson

Contribution No. 210, we included among the giant stars those of exceptionally great intrinsic brightness as well as ordinary giants, and those of brightness intermediate between giants and dwarfs, in order to increase the range in absolute magnitude. Within this range the correlation curve between velocity and absolute magnitude was assumed to be continuous, a conclusion which appeared to be justified from a study of the average radial velocity as a function of absolute magnitude. In this way the systematic effect of the accidental errors in the absolute magnitudes was much reduced.

As an illustration we may refer to the giant stars of spectral types K4 to K9 discussed in Contribution No. 210. The dispersion in observed absolute magnitude was found to be so small that it seemed doubtful whether it would be possible by any means to determine a relationship between velocity and absolute magnitude. Accordingly this group was combined with that of types K0 to K3 and a larger range in intrinsic brightness was obtained in this way.

The correction derived by Professor Eddington and Miss Douglas is strictly applicable to the τ-components of the proper motions, and a similar but smaller correction should be applied to the space velocities used in Contribution No. 210. The tangential velocity T, however, is the total proper motion converted into linear measure and is not comparable directly with the τ-component, since a large portion of T is due to parallactic motion for which there is no dependency upon intrinsic brightness, if the solar motion is supposed to be independent of absolute magnitude.

The value of 15·5 km. found by Professor Eddington and Miss Douglas for the linear τ-component corresponding to the median absolute magnitude of giant stars of types G8 to K2 may be compared with that of about 18·5 km. derived by us from radial velocities. The difference is probably due to the inclusion of a large number of stars of very small proper motion. The effect of this selection is especially marked among these spectral subdivisions, and a correction was applied by us in our investigation in an attempt to eliminate its influence.

Note on the Effect of Astigmatism on Star-Discs.

By Arnold Buxton, M.A.

I. *Recapitulation.*

In previous papers on this subject * the relative intensities in the diffraction disc-image of a star in a telescope have been determined in the following cases:—

(*a*) The ideal image. (Airy spurious disc.)	Positions of the dark rings given by the roots of $\frac{2}{z}J_1(z)=0$;	Intensity formula $I=\left[\frac{2}{z}J_1(z)\right]^2$ where $J_1(z)$ is the Bessel Function of 1st order.
	light rings by roots of $J_2(z)=0$	,, $J_2(z)$,, ,, 2nd ,,

* "Star-Discs," Professor A. E. Conrady, *Monthly Notices R.A.S.*, **79**, 575.
"Star-Discs," A. Buxton, *ibid.*, **81**, 547.
"Star-Discs (Coma)," Dr. L. Martin, *ibid.*, **82**, 310.

(*b*) The disc-image in a plane parallel to the ideal focal plane, and therefore out-of-focus.	Light and dark rings given by the maxima and zeros of the intensity formula.	$I=\left[\int_0^1 J_0(r\mu)\cos cr^2\pi dr^2\right]^2+\left[\int_0^1 J_0(r\mu)\sin cr^2\pi dr^2\right]^2$ where c denotes the number of wave-lengths out of focus.
(*c*) The disc-image in the presence of primary spherical aberration.	Do.	$I=\left[\int_0^1 J_0(r\mu)\cos cr^4\pi dr^2\right]^2+\left[\int_0^1 J_0(r\mu)\sin cr^4\pi dr^2\right]^2$ where c denotes the number of wave-lengths of the lead of the marginal over the paraxial ray.
(*d*) The disc-image under a combination of (*c*) and (*b*).	Do.	$I=\left[\int_0^1 J_0(\mu r)\cos(cr^4+c_1r^2)\pi dr^2\right]^2+\left[\int_0^1 J_0(\mu r)\sin(cr^4+c_1r^2)\pi dr^2\right]^2$ c and c_1 having the same meanings as in (*c*) and (*b*) respectively. The best disc-image occurs when $c=-c_1$ and its plane is half-way between the marginal and paraxial focal planes.
(*e*) The disc-image in the presence of pure coma, the other aberrations being supposed absent.	Do.	$I=\left[\int_0^1 J_0(\mu r+\mu_1 r^3)\pi dr^2\right]^2$ where μ_1 varies as the length of the coma figure.

The positions of the first dark rings, important from the point of view of resolution, have been determined numerically or otherwise in all the above cases for different values of the constants. The loci of the dark rings for the case of coma may be shown to be given by the equation

$$\int_0^1 J_0\left(\mu r+\frac{r}{f}\sqrt{(y-\mu r^2)^2+x^2}\right)\pi dr^2=0$$

where the coma axis is the axis of y, the point where the principal ray meets the coma axis the origin, and the plane of the axes perpendicular to the optical axis. When the coma figure is away from the Petzval surface, *i.e.* when coma is combined with spherical aberration or out-of-focus or both, the intensity formulæ may again be obtained as in (*d*), replacing $J_0(\mu r)$ by $J_0(\mu r+\mu_1 r^3)$.

II. *The Distribution of Light in the Disc-image when Astigmatism is Present.*

So far as the monochromatic aberrations are concerned, however, it remains to investigate the effect of astigmatism. In the first place, it will be assumed that astigmatism only is present, and that it is of the first type, where the bundle of rays are symmetrical in both the sagittal and tangential planes. The results, however, may be taken to hold for the other types, for Czapski has shown that even when the tangential focal line is not perpendicular to the principal ray the error introduced by taking it so is only slight.

In fig. 1 MF_s and M_1F_t represent the sagittal and tangential focal lines respectively; F_s, F_t being the corresponding foci. OP_o is the principal oblique ray, and P_o is the focus of the small oblique ray pencil on the Petzval surface. The marginal tangential rays will focus at F_s,

and the marginal sagittal rays at F_t. These and the other marginal rays will intersect the Petzval surface approximately in an ellipse AXB. Now consider the path of a ray from a point P on the aperture co-ordinates o, $r \sin E$, $r \cos E$. The ray will intersect MF_s between M and F_s and M_1F_t below F_t, and will intersect the perimeter of the ellipse on the Petzval surface in Q, whose co-ordinates by similar triangles are $f, -r \sin E df/(f-df)$, $-3r \cos E df/(f-3df)$ where $F_tP = df$ (so that $F_sP = 3df$) and $OP_o = f$.

Writing $k = df/f$, the intensity at a point Q_1 near to Q distant y_1 from it is proportional to

$$\left[\int_0^1\int_0^{2\pi} \cos\{(f^{-1}ry_1 \sin E) + f^{-1}kr^2(\sin^2 E + 3\cos^2 E)\}\pi d E dr^2\right]^2$$
$$+\left[\int\int_0^{2\pi} \sin\{(f^{-1}ry_1 \sin E) + f^{-1}kr^2(2 + \cos 2E)\}\pi d E dr^2\right]^2.$$

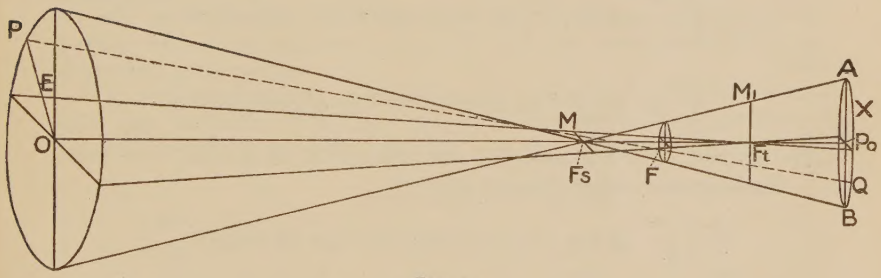


FIG. 1.

Similarly the intensities on the sagittal and tangential focal lines can be determined. For the ray intersects the sagittal focal line, for example, in the point $(f - 3df,\ 2r \sin E df/(f-df),\ o)$, and therefore the intensity near this point is

$$\left[\int_0^1\int_0^{2\pi} \cos\{f^{-1}(ry_1 \sin E - 2kr^2 \sin^2 E)\}\pi d E dr^2\right]^2$$
$$+\left[\int_0^1\int_0^{2\pi} \sin\{f^{-1}(ry_1 \sin E - 2kr^2 \sin^2 E)\}\pi d E dr^2\right]^2.$$

Along the tangential focal line the intensities are similarly given by

$$\left[\int_0^1\int_0^{2\pi} \cos\{f^{-1}(ry_1 \cos E + 2kr^2 \cos^2 E)\}\pi d E dr^2\right]^2$$
$$+\left[\int_0^1\int_0^{2\pi} \sin\{f^{-1}(ry_1 \cos E + 2kr^2 \cos^2 E)\}\pi d E dr^2\right]^2.$$

The smallest and most symmetrical concentration of light, however, occurs midway between the tangential and sagittal focal lines at F; it can be shown that the distribution in this plane through F parallel to AXQB is circular, and, as Southwell points out, it is this image which

the eye unconsciously selects. Hence it will be useful to study more closely the comparative values of the intensities in this circular concentration The central intensity in the Airy spurious disc will again be taken to represent the standard intensity and to be equal to unity. The ray intersects the circular area in the point whose co-ordinates are clearly

$$f-2df, \quad r\sin \mathrm{E}df/(f-3df), \quad -r\cos \mathrm{E}df/(f-3df),$$

giving the intensity formula as

$$\left[\int_0^1\int_0^{2\pi} \cos\{f^{-1}(ry_1 \sin \mathrm{E} - kr^2 \cos 2\mathrm{E})\}\pi d\mathrm{E}dr^2\right]^2 + \left[\int_0^1\int_0^{2\pi} \sin\{f^{-1}(ry_1 \sin \mathrm{E} - kr^2 \cos 2\mathrm{E})\}\pi d\mathrm{E}dr^2\right]^2.$$

The first integral can be written

$$\left[\int_0^1\int_0^{2\pi} \cos(ry_1 f^{-1} \sin \mathrm{E}) \cos(kr^2 f^{-1} \cos 2\mathrm{E})\pi d\mathrm{E}dr^2\right]^2,$$

since

$$\int_0^{2\pi} \sin(ry_1 f^{-1} \sin \mathrm{E}) \sin(kr^2 f^{-1} \cos 2\mathrm{E})\pi d\mathrm{E}dr^2 = 0.$$

[The integral from 0 to 2π of an odd power of sin E is zero.]

Similarly the second integral is

$$\left[\int_0^1\int_0^{2\pi} \cos(ry_1 f^{-1} \sin \mathrm{E}) \sin(kr^2 f^{-1} \cos 2\mathrm{E})\pi d\mathrm{E}dr^2\right]^2.$$

These integrals have been numerically computed for different lengths of the astigmatic figure with variations in $\frac{y_1}{f}$ for any one length; *i.e.* taking k/f as $\frac{1}{4}\lambda$, the integrals have been computed for values of $\frac{y_1}{f}$ 0°, 40°, 80°, etc., for $f/15$. The computation was also effected for the value $\frac{1}{2}\lambda$ for k/f, which corresponds to four times the maximum tolerance of the astigmatic length * for the $f/15$ ratio.

The central intensities are given by the formula

$$\mathrm{I} = \left[\int_0^1\int_0^{2\pi} \cos(kr^2 f^{-1} \cos 2\mathrm{E})\pi d\mathrm{E}dr^2\right]^2 = \left[2\pi^2 \int_0^1 \mathrm{J}_0(kr^2 f^{-1})dr^2\right]$$

since

$$\int_0^1\int_0^{2\pi} \sin(kr^2 f^{-1} \cos 2\mathrm{E})\pi d\mathrm{E}dr^2 = 0.$$

Compare this with the standard central intensity of the Airy spurious disc,

$$\mathrm{I}' = \left[\int_0^1 \mathrm{J}_0(kr^2 f^{-1})dr^2\right]^2.$$

* The astigmatic tolerance may be given as half the focal range due to the Rayleigh limit of a quarter of a wave-length difference of phase on either side of the ideal focus.

This integral has been computed for $k/f = \frac{1}{4}\lambda$ and $\frac{1}{2}\lambda$. For the first value the central intensity is 66 per cent. that of the Airy spurious disc, and for the second it is as low as 18·4 per cent. of the standard. Moreover, it is clear that the intensity of the light in the astigmatic disc-images is comparatively stronger in the outer part of the pattern and weaker in the middle. The radii of the rings of minimum intensity are wider than in the case of equal amounts of the best combination of the other aberrations. At the Rayleigh limit the central intensity is 90 per cent. of that of the Airy spurious disc, but the corresponding diffraction pattern can still be shown to be inferior to that of $\frac{\lambda}{4}$ spherical aberration combined with $\frac{\lambda}{4}$ out of focus, though better than in the cases when these amounts are taken separately. Consequently the presence of astigmatism can be said to detract from the good quality of the image, and as the length of the astigmatic figure increases the disc-image rapidly deteriorates. The resolution of double stars, which depends on the radius of the first dark ring, is adversely affected in the presence of astigmatism, even when the eye has chosen the disc-image of smallest dimensions.

The following table of amplitudes and intensities illustrates numerically the character of the disc-images in the presence of amounts of astigmatism double and four times the astigmatic tolerance for $f/15$.

k/f.	W_1.	Amplitude (cosine integral).	Amplitude (sine integral).	Intensity.
$\frac{1}{4}\lambda$	0°	·817	0	·66
	40	·698	·016	·49
	80	·630	·058	·40
	120	·537	·125	·30
	240	·195	·188	·07
	360	·014	·085	·007
	480	·056	·08	·009
	600	·155	·141	·04
	720	·154	·093	·03
	840	·117	·063	·02

k/f.	W_1.	Amplitude (cosine integral).	Amplitude (sine integral).	Intensity.
$\frac{1}{2}\lambda$	0°	·429	0	·184
	40	·420	·013	·176
	80	·382	·047	·148
	368	−·060	·127	·020

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Occultation of Aldebaran, 1923 *January* 27, *observed at the Radcliffe Observatory, Oxford.*

(*Communicated by Dr. Rambaut, F.R.S.*)

The following observations of the occultation of Aldebaran on 1923 January 27 were made by Mr. J. G. Balk.

The resulting G.M.T.'s will be found to be in close agreement with Professor Turner's, given in *M.N.* for 1923 April.

Phenomenon.	Sidereal Time Oxford (Radcliffe).	Greenwich Mean Time.
	h m s	h m s
Aldebaran dis.	8 54 46·3	12 34 47·1 (*a*)
,, reapp.	9 49 45·1	13 29 36·9 (*b*)

Telescope—Barclay 10-inch, power 90.

Longitude $5^{m}\ 2^{s}\cdot6$ W.; latitude $+51^{\circ}\ 45'\ 35''$.

Observer's remarks:—

(*a*) Difficult, owing to noises from trains.

(*b*) Very diffused; clouds passing.

Radcliffe Observatory, Oxford:
1923 *June* 6.

Erratum.

P. 350, *for* mass 2×10^{33} grams *read* mass 2×10^{30} grams.